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Optimal Control Problems under Uncertainty

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ABSTRACT

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Declaration of Authorship

I, Boyan Kolev STEFANOV, declare that this thesis titled, “Optimal Control Problems under Uncertainty” and the work presented in it are my own. I confirm that:

- The results presented in this thesis are the result of my own work or in collaboration with my co-authors.
- Where I have consulted the published work of others, this is always clearly attributed.
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"Nothing takes place in the universe in which some rule of maximum or minimum does not appear. "

Leonhard Euler (1707-1783)

Introduction

Optimal control theory is a mathematical discipline that focuses on finding a control law for a dynamical system over time to optimize a certain objective, typically expressed as a cost function. It has applications across various fields, including economics, engineering, environmental sciences, and management. Key concepts in optimal control theory include *state variables* and *control variables*, *cost functions*, *differential equations*, and *optimization*.

The roots of optimal control theory can be traced back to the calculus of variations, a field of mathematics dating back to Newton and Leibniz in the 17th century. It deals with finding the extrema (maximum or minimum values) of functionals, which are mappings from a set of functions to the real numbers. Functionals are often expressed as integrals involving functions and their derivatives. It is often considered to have its origins in 1696 with the work of Johann Bernoulli and the Brachistochrone problem. Johann Bernoulli posed the Brachistochrone problem in 1696, which is considered one of the earliest problems in the calculus of variations, it was posed in the journal "Acta Eruditorum," which was one of the first scientific journals of the time. The problem involves finding the curve of quickest descent, which is, the path along which a particle will move under the force of gravity from one point to another in the shortest time. This problem attracted the attention of some of the most prominent mathematicians of the time, including Leibniz, Newton, and Bernoulli's own brother, Jakob Bernoulli. The solution to the Brachistochrone problem was a significant moment in the history of mathematics, as it was one of the first times a mathematical solution was found for a problem of optimizing a quantity - in this case, time. It laid the groundwork for the development of the calculus of variations as a formal mathematical discipline. A problem formulated in terms of calculus of variation, can be written as follows: Given a functional

$$J[x] = \int_a^b L(x(t), \dot{x}(t), t)\, dt$$

where $x(t)$ is a function that maps from $[a, b] \subset \mathbb{R}$ to $\mathbb{R}$, and $\dot{x}(t)$ is the derivative of x with respect to t. The task is to find the function $x(t)$ that extremizes (minimizes or maximizes) the functional $J[x]$.

Over the next century, the calculus of variations grew as a field of study, with contributions from mathematicians like Euler and Lagrange. Euler's significant contribution in the calculus of variations was the development of the Euler - Lagrange equation, a fundamental equation to find the function that minimizes or maximizes a functional. The equation is expressed as:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0,$$

where L is the Lagrangian. Euler laid the foundation for the variational principles, demonstrating how extremal values of functionals are found by solving differential equations.

Lagrange developed Lagrangian mechanics, a reformulation of classical mechanics, introduced in his work "Mécanique Analytique" (1788). Lagrange introduced the concept of generalized coordinates and formulated the principle of least action. This principle states that the path taken by a physical system between two states is the one for which the action is minimized.

Throughout the 18th and 19th centuries, the focus was largely on developing mathematical techniques for optimizing functionals, setting the stage for the formal establishment of optimal control theory in the 20th century, which extends this framework to include functionals of functions that are subject to dynamic constraints, thus broadening the scope of problems that can be addressed in an optimization context.

Given controlled dynamics, the following is an optimal control problem formulation:

$$\begin{aligned} &\text{Maximize: } J[u(t)] = \int_{t_0}^{t_f} L(x(t), u(t), t)\, dt \\ &\text{Subject to: } \begin{cases} \dot{x}(t) = f(x(t), u(t), t) & \text{(System Dynamics)} \\ u(t) \in U & \text{(Control Constraints)} \\ x(t_0) = x_0 & \text{(Initial Condition)} \\ x(t_f) = x_f & \text{(Final State Condition, optional)} \end{cases} \end{aligned} \tag{P}$$

where $x(t)$ represents the state of the system at time t, $u(t)$ represents the control input at time t, U is the closed set of all admissible controls, $f(x, u, t)$ is the function defining the system's dynamics, and $L(x, u, t)$ is the cost function. The goal is to find the control function $u^*(t)$ that minimizes $J[u]$ over the interval $[t_0, t_f]$, adhering to the specified dynamics and constraints.

The Pontryagin Maximum Principle, developed by Russian mathematician Lev Pontryagin in the late 1950s, is a significant advancement in the field of control theory (cf. Pontryagin et al., 1962). It is a cornerstone of optimal control theory that provides necessary conditions for an optimal control process.

The Principle provides a method to find the best control strategy for a given system. It does this by introducing an auxiliary function called the Hamiltonian, which combines the system's dynamics with the objective we want to optimize.

The principle informally says that for a control strategy to be optimal, it must, at each instant in time, adjust the control variable such that the Hamiltonian is maximized (or minimized). This is like saying, "At every moment, choose your control action in such a way that it gives you the best immediate advantage, considering both your current state and your goal."

Formally the principle states:

Consider the optimal control problem (P). The Pontryagin Maximum Principle states that if $u^*(t)$ is an optimal control function on the interval $[t_0, t_f]$, with corresponding optimal trajectory $x^*(t)$, there exists a non-trivial, absolutely continuous function $\psi(t)$, called the adjoint variable or costate, such that:

The adjoint variable $\psi(t)$ satisfies the adjoint equation:

$$\dot{\psi}(t) = -\frac{\partial \mathcal{H}}{\partial x}(x^*(t), u^*(t), \psi(t), t),$$

where $\mathcal{H}$ is the Hamiltonian defined as:

$$\mathcal{H}(x, u, \psi, t) = \psi \cdot f(x, u, t) + L(x, u, t)$$

For almost every t, the Hamiltonian H is maximized (or minimized for minimization problems) with respect to u along the optimal trajectory:

$$\mathcal{H}(x^*(t), u^*(t), \psi(t), t) = \max_{u \in U} \mathcal{H}(x^*(t), u, \psi(t), t),$$

where U is the set of all admissible controls.

At the same time, Richard Bellman, an american mathematician, developed the method of dynamic programming, a recursive optimization strategy that breaks down multi-period planning problems into simpler steps at different points in time (cf. Bellman, 1957). The Principle of Optimality, formulated by Bellman, is a foundational concept in dynamic programming and optimal control theory, providing a critical framework for solving optimization problems where decisions need to be made sequentially over time. Mathematically, the Bellman Principle of Optimality can be expressed as follows:

For a dynamic system with a state $x(t)$ at time t and a control strategy $u(t)$ over a time horizon $[t_0, T]$, *the value function* $V^*(x(t), t)$ is defined as:

$$V^*(x(t), t) = \min_{u(\tau)} \left\{ \int_t^T g(x(\tau), u(\tau), \tau) d\tau + h(x(T)) \right\},$$

where g is the running cost function, h is the terminal cost function, and τ ranges from t to T. The principle states that if $u^*(t)$ is an optimal control strategy from t_0 to T, then for any intermediate time t_1, $u^*(t)$ from t_1 to T must also be optimal for the system starting in state $x(t_1)$. This implies that the decision at any time t depends only on the state at that time and not on the prior path taken to reach that state.

In other words it states that an optimal policy or strategy has the property that, regardless of the initial state and initial decision, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. In simpler terms, it means that if you have found an optimal path (or strategy) from some initial state to a goal state, then every subpath (or sub-strategy) of this path, starting from any point along the path, must also be optimal.

This principle allows the decomposition of a complex dynamic optimization problem into smaller, simpler subproblems, which can be solved independently, and the solutions can be combined to solve the original problem. It is particularly powerful in the context of discrete-time systems, where it leads to a recursive algorithmic structure, often referred to as "dynamic programming".

Natural processes, engineering, manufacturing, economics, healthcare, information systems, and social and behavioral processes all face inherent uncertainties. Environmental conditions like weather patterns, geological events, and ecological dynamics are rife with variability, posing challenges for prediction and control. Engineering and manufacturing design and optimization are complicated by material properties, production processes, and market demand. Economic and financial systems are inherently uncertain, driven by market fluctuations, geopolitical events, and unforeseen crises. Healthcare faces uncertainties in biological systems, treatment effectiveness, and disease spread, while information systems and technology face uncertainties from network congestion, software bugs, and cybersecurity threats. Social and behavioral processes, such as human decision-making and interactions, are also influenced by uncertainties from individual preferences, societal dynamics, and unforeseen events. Political, cultural, and economic shifts contribute to the complex and unpredictable nature of social systems. This introduction aims to illuminate the omnipresence of uncertainty and its multifaceted relationship with various real-world processes, suggesting that uncertainty is a fundamental aspect of many real-world phenomena and plays a complex role in shaping and influencing various processes.

The traditional optimal control paradigm, which focuses on optimizing system behavior, faces new challenges in developing strategies resilient to unforeseen disturbances or imprecise knowledge. This leads to a specialized branch of optimal control theory that considers uncertain environments. Including a disturbance term in the objective function adds complexity to finite and infinite time horizon optimal control problems, as disturbances represent external factors that affect system behavior and cost.

The significance of addressing uncertainty is highlighted by the number of mathematical concepts developed to manage it. While this research doesn't explore these concepts, notable examples include:

Probability Theory plays a pivotal role in modeling and analyzing uncertainties, essential in optimal control and decision-making. The foundational works of notable mathematicians such asMoivre, 1718, Gauss, Laplace, 1812, and Kolmogorov, 1956 have been instrumental in shaping this field.

Stochastic processes, which model dynamic systems with random fluctuations, have been extensively studied by mathematicians like Doob, 1953 and Taylor and Karlin, 1975. Stochastic processes model systems whose behavior evolves over time under random influences. They are characterized by variables that change stochastically, often described by stochastic differential equations. Stochastic processes are used to represent dynamic systems affected by noise or other random factors, allowing for the study of their long-term behavior and the impact of uncertainties on system performance.

Stochastic Differential Equations (SDEs) are differential equations in which one or more of the terms is a stochastic process, often representing noise or other random influences. SDEs are used to model the dynamics of systems affected by random fluctuations, providing a means to analyze and predict the behavior of such systems under uncertainty, as explored by Øksendal, 2003.

Statistics, a key tool for analyzing uncertain data, finds its roots in the contributions of statisticians such as Fisher, 1925 and Neyman, 1934. Ronald A. Fisher, made contributions to the development of statistical methods, including the introduction of maximum likelihood estimation (MLE). While Fisher's work laid the foundation for likelihood-based inference, including hypothesis testing through methods like the likelihood ratio test, the likelihood ratio test was further developed by statisticians like Jerzy Neyman and Egon Pearson. Neyman's work, particularly the Neyman-Pearson lemma, laid the groundwork for systematic hypothesis testing.

Fuzzy Logic, introduced by Zadeh, 1965 extends classical logic by introducing degrees of truth. It's used in situations where information is ambiguous, imprecise, or lacking in certainty. Fuzzy logic enables the handling of uncertainty by allowing for intermediate values between 'completely true' and 'completely false', thus providing a more nuanced approach to decision-making and control in uncertain environments.

Information theory, pioneered by Shannon, 1948, introduces measures like entropy to quantify uncertainty in communication systems and data analysis. Shannon's main result is the formulation of entropy as a measure of uncertainty or information content in a probabilistic system. Shannon's entropy measures the average amount of surprise or uncertainty associated with the possible outcomes of a random variable. It quantifies the information content in a probability distribution, with higher entropy indicating higher uncertainty. If all outcomes are equally likely, the entropy is maximized, representing maximum uncertainty. Conversely, if one outcome is certain (probability 1) and others impossible (probability 0), the entropy is minimized, representing no uncertainty.

Monte Carlo Methods are computational algorithms used for simulating random processes. Monte Carlo methods are applied in optimal control to estimate the behavior of complex systems under uncertainty by running simulations with random inputs. This approach is particularly useful for assessing the impact of uncertainty and for systems that are analytically intractable. Nicholas Metropolis and Stanislaw Ulam are pioneers in this field.

Robust optimization, seeking solutions resilient to uncertainty, draws inspiration from Ben-Tal and Nemirovski, 2009 seminal work and Bertsimas and Sim, 2004 contributions. Robust Optimization approach involves formulating optimization problems that remain effective under various scenarios of uncertainty. It typically focuses on designing solutions that can withstand worst-case scenarios or a range of uncertain parameters. Robust optimization is crucial in ensuring that the solutions to control problems are not overly sensitive to assumptions or unknown variables.

Designing and implementing robust controls for such systems has the advantage of ensuring acceptable outcomes in a range of circumstances, while decisions entirely based on expected utility would generally lead to inferior performance if the realization of the uncertain quantities deviates significantly from the expectation. Furthermore, sometimes it is not feasible to assign probabilities to the various scenarios (Knightian uncertainty) which renders the expected utility approach inapplicable. An alternative is to seek to minimize the loss associated with the worst outcome if a certain decision rule is applied, which naturally leads to min-max decision criteria. Robust control methods have found broad applications in engineering, including aerospace (see, for example, Chen and Hu, 2007, and Durham and Lawrence, 2007), electrical (Zhou and Doyle, 1997, and Wilamowski and Irwin, 2011) and industrial (Antsaklis and Michel, 2006, Doyle, Francis, and Tannenbaum, 2009, and Petersen and Savkin, 2003), but also in various economic problems, notably in monetary theory and policy. Examples include Rozenov, 2016, Onatski and Williams, 2003, Giannoni, 2007 and J. Dennis and Soderstrom, 2009, among others. In economic applications, Hansen and Sargent, 2008 approach to robustness is motivated by the fact that often decision makers work with models that are only approximations of the true model that generates the data. Faced with the problem of misspecification, the decision maker seeks a rule that will perform well across all models that satisfy a relative entropy constraint. This entropy constraint is usually converted into a penalty term in the objective function to capture the preference for robustness. A related interpretation is that of a game against a hypothetical malevolent agent (nature) which chooses the disturbances such as to maximize the loss that the policymaker is trying to minimize.

The game theory is fundamental to this setting because it offers a framework for assessing situations in which numerous agents are involved in decision-making, each player's result is dependent on the decisions made by others, and one agent's actions affect or influence others. Dynamic games, have been widely used to model strategic interactions when the players have different goals. While historically they have been largely associated with pursuit-evasion games, recent applications focus more on attenuation of disturbances in controlled uncertain systems. The theory has been shaped by the works of Neumann and Morgenstern, 1944 and Nash, 1950. Central to the analysis of these games is the concept of equilibrium, where players' strategies reach a state of equilibrium, and no player has an incentive to unilaterally deviate from their chosen strategy. In the context of the infinite time horizon, various notions of equilibrium, such as perfect Nash equilibrium, Stackelberg equilibrium, weakly overtaking equilibrium, among others play a crucial role in characterizing stable and sustainable strategies that persist over time.

Different approaches to obtaining robust controls have been developed in the literature, both in the frequency and time domains. In the time domain, the min-max formulation provides a natural association with a two-player non-cooperative zero-sum game (Başar and Bernhard, 1991). In this setup, one of the players aims to minimize the loss, whereas the other player (nature) aims to maximize it. Thus, the solution to the min-max optimal control problem, when it exists, represents the saddle point equilibrium of the game, which is also the Nash equilibrium.

Dynamic games can take place in continuous or discrete time and can be defined on finite or infinite time horizon (Başar and Olsder, 1999). Among them, linear quadratic unconstrained games are perhaps the most studied class (Engwerda, 2005). In many practical circumstances, however, the assumption that the players' actions are unconstrained is not appropriate. .

Chapter 1 applies the dynamic game perspective to the problem of robustness. In particular, it considers two-person non-cooperative linear-quadratic (LQ) games in continuous time on an infinite-time horizon. This approach is in line with the framework proposed by Başar and Bernhard, 1991, which utilizes game theory to formalize the worst-case design problem. While LQ games have been well-studied in the literature (e.g., Başar and Olsder, 1999; E. Dockner and Sorger, 2012; Engwerda, 2005), the infinite-time horizon case is usually presented in a context where no constraints are imposed on the control variables. This drawback is particularly significant in real-world scenarios where control actions are inherently bound by certain limitations. An example of such a constraint is the zero lower bound on nominal interest rates. The chapter aims to bridge this gap by exploring scenarios where control variables are subject to realistic constraints, thereby providing a more comprehensive understanding of LQ games in infinite-time horizons.

The chapter's primary contribution lies in establishing a sufficient condition for a saddle point, which also corresponds to the Nash equilibrium, in an infinite-time horizon LQ game, especially under constraints on the control actions of the minimizing player. We demonstrate the existence of a compact neighborhood around the origin within the state space, where these constraints are not active. The existence of such a neighborhood, as detailed in Goebel and Subbotin, 2007, is previously established for a specific linear-quadratic optimal control problem. Utilizing this neighborhood, we transform the infinite-time horizon optimal control problem faced by the minimizing agent into a finite-time horizon problem, which is then solvable using standard methods. Although our results are framed within the context of saddle points (Nash equilibria), they are equally applicable to Stackelberg game formulations, which are often more suitable for certain economic models. This applicability stems from the alignment of stationary feedback Nash and Stackelberg equilibria in games characterized by orthogonal reaction functions, as outlined in Rubio, 2006.[1]

Definition 0.1 (Stackelberg equilibrium, Chen and Cruz, 1972)**.** *Given a two-person game, where Player 1 wants to minimize a cost function $J_1(u_1, u_2)$ and Player 2 wants to minimize a cost function $J_2(u_1, u_2)$ by choosing u_1, u_2 from admissible strategy sets U_1 and U_2, respectively, then the control pair (u_1^*, u_2^*) is called a* Stackelberg equilibrium *with Player 2 as leader and Player 1 as follower if for any u_2 belonging to U_2 and u_1*

[1] Differential games with orthogonal instantaneous reaction functions are characterized by the independence of the first derivatives of the cost functions and dynamic equations with respect to each player's controls from the controls of the other player (see Rubio, 2006, Definition 2.4).

belonging to U_1

$$J_2(u_1^*, u_2^*) \leq J_2(u_1'(u_2), u_2),$$

where

$$J_1(u_1'(u_2), u_2) = \min_{u_1} J_1(u_1, u_2)$$

and

$$u_1^* = u_1'(u_2^*).$$

In Section **??** of Chapter 1, we illustrate the proposed approach using a simple New Keynesian monetary policy model, formulated in continuous time. This model serves as a practical example to demonstrate the application and effectiveness of our theoretical findings in a real-world economic context.

In Chapter 2, the focus is on discrete-time linear-quadratic games over an infinite horizon, again with a special emphasis on scenarios where constraints are applied to the actions of the agent seeking minimization.Although sharing similarities with its continuous-time counterpart, discussed in the previous chapter, the discrete-time case presents unique challenges that warrant distinct analysis. The adoption of discrete-time systems becomes particularly pertinent when the nature of the problem under scrutiny is inherently discrete, aligning with the discrete nature of the control actions involved. Central to our approach is Bellman's principle of optimality, which we demonstrate to be applicable across a broad spectrum of min-max control problems.

The contributions of this chapter include a min-max theorem for the unconstrained linear-quadratic problem and an exploration of the asymptotic behavior of system trajectories under control. One of our main findings is the establishment of an equivalence between the infinite-horizon problem and a corresponding finite-horizon problem. We show that, in cases where the system is stabilizable, there exists a neighborhood near the origin where control constraints become non-binding. Consequently, once the state enters this neighborhood, the solution for the constrained problem aligns with that of the unconstrained problem, effectively reducing it to a finite-dimensional issue solvable through numerical methods. The main contribution in Chapter 2 presents sufficient optimality conditions for the linear-quadratic discrete-time game.

While some of the results obtained in the chapter are known, the presented proofs are novel. The practical application of our approach is illustrated through a model detailing the short-period dynamics of an F-16 aircraft.

Chapter 3 delves into the complexity of deriving optimality conditions of Pontryagin's maximum principle type (as initially developed in Boltyanskij, 1978). The optimization problem under consideration falls within the class of min-max type problems, where the objective is to minimize the maximum adverse impact of disturbances.

Several authors have significantly contributed to this field. For instance, Fudenberg and Tirole, 1991 present a range of results in game theory, focusing on equilibria in discrete-time games. Their approach, grounded in dynamic programming and backward induction, is used for formulating and solving these game problems. Osborne and Rubinstein, 1994 delve into Nash equilibria in discrete-time games, employing best response dynamics and mathematical tools like fixed-point theorems to understand strategic behavior in a variety of game-theoretic settings.

A pivotal outcome of the investigation in this chapter is the establishment of a necessary optimality condition, structured along the lines of Pontryagin's maximum principle. The derivation leverages a crucial result from a recent study by Aseev, Krastanov, and Veliov, 2017, denoted as Theorem 2.2, which provides a local maximum condition for the Hamiltonian function.

Additionally Chapter 3 focuses on a specific subset of discrete-time games within dynamical games, specifically optimal control problems without disturbances, highlighting that optimal control problems are a partial manifestation within the broader dynamical game canvas.

The central result of this section, Theorem 3.17, provides under suitable assumptions a novel sufficient optimality condition. The proof relies on a well-established relationship between the objective function of the optimal control problem and the corresponding Hamiltonian function, as explained in Proposition 3.16. This crucial interrelation was deduced utilizing a definition in Aseev, Krastanov, and Veliov, 2017, where as previously explain the adjoint variable – critical for the maximum condition for the Hamiltonian in Pontryagin's Principle – is explicitly defined for each given optimal process.

Contents

Declaration of Authorship i

Dissemination ii

Acknowledgements iv

Introduction v

1 Continuous-Time Linear-Quadratic Game 1
- 1.1 Formulation of the Problem . . . 1
- 1.2 Approximation of the Constrained Game . . . 3
  - 1.2.1 Solution to the Unconstrained Game . . . 3
  - 1.2.2 Constraint-Free δ-Neighborhood . . . 4
  - 1.2.3 Introducing a Finite-Time Horizon Game . . . 5
- 1.3 Sufficient Optimality Conditions . . . 5

2 Discrete-Time Linear-Quadratic Game 7
- 2.1 Formulation of the Problem . . . 7
- 2.2 Preliminaries . . . 10
- 2.3 Approximation of the Constrained Game . . . 12
- 2.4 Sufficient Optimality Conditions . . . 14

3 Discrete-Time Dynamic Game 16
- 3.1 Formulation of the Discrete-Time Game . . . 16
- 3.2 Preliminaries . . . 20
- 3.3 Necessary Optimality Condition . . . 21
- 3.4 Discrete-Time Optimal Control Problem (OCP) . . . 23
  - 3.4.1 Formulation of the Optimal Control Problem . . . 23
  - 3.4.2 The Objective-Hamiltonian Relation . . . 25
  - 3.4.3 Sufficient Optimality Condition . . . 26

Conclusion 28

Bibliography 29

To all those who have taught me

Chapter 1

Continuous-Time Linear-Quadratic Game

"In everything, there is a share of everything "

Anaxagoras (c.500 - c.428 BC)

1.1 Formulation of the Problem

Let us consider a specific scenario where a real number $t_0 \geq 0$ and an initial state vector $x_0 \in \mathbb{R}^n$ are fixed. In this setting, we study a class of continuous non-cooperative linear-quadratic differential games. The dynamics of the game is governed by the following linear differential equation:

$$\dot{x}(t) = Ax(t) + B_u u(t) + B_v v(t), \quad x(t_0) = x_0, \tag{1.1}$$

where A, B_u, and B_v represent matrices with dimensions $n \times n$, $n \times m_u$, and $n \times m_v$ respectively. The vectors $x(t)$, where $t \in [t_0, +\infty)$, and x_0 signify the system state at time t, and the initial state at time t_0, respectively.

The players, referred to as the first and second players, determine their actions (or controls) via the functions $\mathbf{u}$ and $\mathbf{v}$, respectively. Both controls can either be open-loop or closed-loop.

An open-loop control $\mathbf{u}$ for the first player is a measurable function $u : [t_0, +\infty) \to \mathbf{U} \subset \mathbb{R}^{m_u}$ with $\int_{t_0}^{\infty} \|u(t)\|^2 \, dt < \infty$. The set $\mathbf{U}$ represents a closed and convex neighborhood of the origin in $\mathbb{R}^{m_u}$. The set of all open-loop controls for the first player is denoted by $\mathcal{U}_{\mathbf{U}}$. In the unbounded case ($\mathbf{U} = \mathbb{R}^{m_u}$), this set is denoted by $\mathcal{U}$. A closed-loop control $u : \mathbb{R}^n \to \mathbb{R}^{m_u}$ is a function $u(x)$, where, if linear, $u(x) = K_u x$, and K_u is an $m_u \times n$ matrix.

It is assumed that the second player, representing disturbances, faces no control constraints. An open-loop control $\mathbf{v}$ for the second player is a measurable function $v : [t_0, +\infty) \to \mathbb{R}^{m_v}$ with $\int_{t_0}^{\infty} \|v(t)\|^2\, dt < \infty$. The set of all such controls is denoted by $\mathcal{V}$. A closed-loop control $v : \mathbb{R}^n \to \mathbb{R}^{m_v}$ is a function $v(x)$, where, if linear, $v(x) = K_v x$, and K_v is an $m_v \times n$ matrix.

Given the dynamics (1.1), we consider the following infinite-time horizon linear-quadratic game:

$$\inf_{\mathbf{u} \in \mathcal{U}_{\mathbf{U}}} \sup_{\mathbf{v} \in \mathcal{V}} J(x_0, t_0, \mathbf{u}, \mathbf{v}), \tag{1.2}$$

where the objective function is defined as

$$J(x_0, t_0, \mathbf{u}, \mathbf{v}) := \int_{t_0}^{\infty} \left(x_{\mathbf{u},\mathbf{v}}^{\intercal}(t) Q x_{\mathbf{u},\mathbf{v}}(t) + u^{\intercal}(t) R_u u(t) - v^{\intercal}(t) R_v v(t) \right) dt.$$

In this formulation, Q is a symmetric, positive semi-definite matrix of dimension $n \times n$, while R_u and R_v are symmetric, positive definite matrices with dimensions $m_u \times m_u$ and $m_v \times m_v$, respectively. The objective of the first player is to minimize the objective function $J(x_0, t_0, \mathbf{u}, \mathbf{v})$, whereas the second player aims to maximize it. The notation $x_{\mathbf{u},\mathbf{v}}(\cdot)$ represents the trajectory corresponding to the control functions $u(\cdot)$ and $v(\cdot)$. The **value function** $V_{\mathbf{U}} : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}$ of the game is defined as:

$$V_{\mathbf{U}}(x_0, t_0) := \inf_{\mathbf{u} \in \mathcal{U}_{\mathbf{U}}} \sup_{\mathbf{v} \in \mathcal{V}} J(x_0, t_0, \mathbf{u}, \mathbf{v}).$$

A control pair $(\mathbf{u}, \mathbf{v})$ is considered **feasible**, if, additionally, the objective function J, as described in (1.2), yields a finite value under it. We set

$$V(x_0, t_0) := V_{\mathbb{R}^{m_u}}(x_0, t_0).$$

Below, we state several propositions that establish some properties of the differential game (1.2) and provide a method for finding its solution.

The following assumption will apply to the remainder of this chapter. It sets the stage for analyzing and solving the differential game by ensuring that the system described by the linear-quadratic framework is stable and behaves in a predictable and controllable manner.

StandingAssumption : (*)

- *The matrix algebraic Riccati equation (1.3), provided below*

$$Q + X B_v R_v^{-1} B_v^{\intercal} X - X B_u R_u^{-1} B_u^{\intercal} X + A^{\intercal} X + X A = 0 \tag{1.3}$$

  with respect to the $n \times n$ matrix X, possesses a symmetric positive definite solution, denoted further by P;

- *All eigenvalues of the matrices*

$$A,\ [A - B_u R_u^{-1} B_u^{\intercal} P],\ [A + B_v R_v^{-1} B_v^{\intercal} P],\ \text{and } [A - B_u K_u + B_v K_v]$$

exhibit negative real parts, where

$$K_u := R_u^{-1} B_u^\intercal P \quad \text{and} \quad K_v := R_v^{-1} B_v^\intercal P.$$

Regarding the information pattern, we assume that both players possess complete knowledge of all parameters governing the dynamics, including the matrices A, B_u, and B_v, as well as those defining the objective function, such as the matrices Q, R_u, and R_v. Additionally, we assume that throughout the entire duration of the game, both players have access to the current state of the system, denoted as $x(t)$ for $t \in [t_0, \infty)$. A similar model is considered in Engwerda, 2005.

1.2 Approximation of the Constrained Game

To address the constrained scenario, the strategy involves demonstrating a correlation between the infinite-horizon issue and a corresponding finite-horizon problem. This approach proves beneficial, particularly when the system is stabilizable. In such cases, it's demonstrated that there exists a neighborhood around the origin in the state space where the control limitations cease to be restrictive. When the state reaches this area, the outcome of the constrained problem aligns with the solution of the unconstrained issue. Consequently, this reduces the problem to a finite-time constrained one, which can be effectively resolved using computational techniques.

The assertion below provides a fundamental property of the system (1.1), it establishes that, for an arbitrary control pair

$$(\mathbf{u}, \mathbf{v}) \in L^2([t_0, \infty), \mathbb{R}^{m_u}) \times L^2([t_0, \infty), \mathbb{R}^{m_v}),$$

the state trajectory converges to the origin as time approaches infinity. Here, $L^2([t_0, \infty), \mathbb{R}^m)$ is defined as the set $\{w : [t_0, \infty) \to \mathbb{R}^m : \int_{t_0}^{\infty} |w(t)|^2 dt < \infty\}$, which forms a Banach space with the norm $\|\mathbf{w}\|_{L^2} = \left(\int_{t_0}^{\infty} |w(t)|^2 dt \right)^{1/2}$.

Proposition 1.1 (Vanishing State Trajectory). *Let the Standing Assumption (*) hold true, let*

$$\mathbf{u} \in L^2([t_0, \infty), \mathbb{R}^{m_u}) \quad \text{and} \quad \mathbf{v} \in L^2([t_0, \infty), \mathbb{R}^{m_v})$$

are arbitrary measurable functions and let $\mathbf{x}_{\mathbf{u},\mathbf{v}}$ *be the corresponding state trajectory. Then,*

$$\lim_{t \to +\infty} x_{\mathbf{u},\mathbf{v}}(t) = \mathbf{0}.$$

1.2.1 Solution to the Unconstrained Game

Initially, let's examine the scenario where the first player's controls are unrestricted. Under suitable technical assumptions, the subsequent proposition provides, feedback controls that act as a **Nash equilibrium** for the differential game (1.2), i.e., it ensures optimality in the absence of unilateral improvements in an infinite-time horizon setting. "Optimality in the absence of unilateral improvements" refers to a situation in a game where a strategy profile (a combination of strategies chosen by each player) is considered optimal if no single player can unilaterally deviate from their current strategy to achieve a better outcome for themselves. In other words, the strategy profile is deemed optimal if, given the strategies chosen by all other players, no individual player can improve their own payoff by independently changing their strategy. The Nash equilibrium is crucial to the system's ability to resist changes or deviations by individual players.

Proposition 1.2 (Solution to the Unconstrained Game)**.** *Let the Standing Assumption (*) hold true and let us define the feedback controls*

$$\bar{u}(x) := -K_u x \in \mathbb{R}^{m_u},\ \bar{v}(x) := K_v x \in \mathbb{R}^{m_v},\ x \in \mathbb{R}^n. \tag{1.4}$$

Then,

$$J(x_0, t_0, \bar{\mathbf{u}}, \mathbf{v}) \le V(x_0, t_0) = J(x_0, t_0, \bar{\mathbf{u}}, \bar{\mathbf{v}}) \le J(x_0, t_0, \mathbf{u}, \bar{\mathbf{v}}) \tag{1.5}$$

for each $\mathbf{u} \in \mathcal{U}$ *and each* $\mathbf{v} \in \mathcal{V}$, *i.e.,* $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ *is a saddle point (Nash equilibrium) for the differential game (1.2) in the unconstrained case when* $\mathbf{U} = \mathbb{R}^{m_u}$ *and*

$$J(x_0, t_0, \bar{\mathbf{u}}, \bar{\mathbf{v}}) = x_0^{\mathsf{T}} P x_0.$$

1.2.2 Constraint-Free δ-Neighborhood

Continuing with our analysis, let us now explore the scenario where constraints are imposed on the control function of the minimizing player. For the subsequent developments, we introduce specific notations. By $\bar{\mathbf{B}}^n$ we denote the closed unit ball centered at the origin (here $\bar{\mathbf{B}}^n \subset \mathbb{R}^n$) and by $\bar{x}_{\bar{\mathbf{u}},\bar{\mathbf{v}}}(\cdot, y, \tau)$ the trajectory induced by (1.1), corresponding to the controls $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ and starting from the point y at the moment of time τ ($\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ are defined by (1.4)). The next proposition establishes the existence of a δ-neighborhood of the origin of $\mathbb{R}^n$ where the specified constraints on the control of the first player are consistently met at all points. Moreover, it is possible to select this neighborhood in such a way that any trajectory beginning from a point within it not only stays within a region where the imposed control constraints are inactive indefinitely but also converges to the origin as time progresses.

Proposition 1.3 (Constraint-Free δ-Neighborhood)**.** *Let the Standing Assumption (*) hold true, and let the matrix*

$$\bar{Q} := Q + PB_u K_u - PB_v K_v$$

be positive definite. Then, there exist positive reals δ_0 *and* δ, *with* $\delta_0 > \delta$, *such that:*

(i) the inclusion $-K_u x \in \mathbf{U}$ *holds true for each* $x \in \delta_0 \bar{\mathbf{B}}^n$;

(ii) for every point $y \in \delta \bar{\mathbf{B}}^n$ *and for every* $\tau \ge 0$, *the inclusion* $\bar{x}_{\bar{\mathbf{u}},\bar{\mathbf{v}}}(t, y, \tau) \in \delta_0 \bar{\mathbf{B}}^n$ *holds true for all* $t \ge \tau$, *and the trajectory* $\bar{x}_{\bar{\mathbf{u}},\bar{\mathbf{v}}}(t, y, \tau)$ *converges to the origin as* $t \to +\infty$.

Remark 1.4. *For the sequel we will fix the parameter δ, as introduced in Proposition 1.3.*

Remark 1.5. *Proposition 1.1 implies the existence of a moment of time $T \geq t_0$, when the state trajectory $x_{\mathbf{u},\mathbf{v}}(\cdot)$, corresponding to an arbitrary admissible control pair $(\mathbf{u}, \mathbf{v})$, enters the above defined δ-neighborhood of the origin, i.e., $x_{\mathbf{u},\mathbf{v}}(T) \in \delta\bar{\mathbf{B}}^n$.*

1.2.3 Introducing a Finite-Time Horizon Game

Let us fix a real number $T > 0$, and consider the following differential game on the finite-time interval $[t_0, T]$

$$V_{\mathbf{U},T}(x_0, t_0) := \inf_{\mathbf{u}\in\mathcal{U}_{\mathbf{U},T}} \sup_{\mathbf{v}\in\mathcal{V}_T} J_T(x_0, t_0, \mathbf{u}, \mathbf{v}), \tag{1.6}$$

subject to (1.1), where

$$J_T(x_0, t_0, \mathbf{u}, \mathbf{v}) := x^{\mathsf{T}}_{\mathbf{u},\mathbf{v}}(T) P x_{\mathbf{u},\mathbf{v}}(T) + \int_{t_0}^{T} \left(x^{\mathsf{T}}_{\mathbf{u},\mathbf{v}}(t) Q x_{\mathbf{u},\mathbf{v}}(t) + u^{\mathsf{T}}(t) R_u u(t) - v^{\mathsf{T}}(t) R_v v(t) \right) dt.$$

In this context, we utilize definitions of admissible controls similar to those in the infinite-time horizon case, employing the subscript T to differentiate between finite and infinite-time horizon case. Consequently, we use $\mathcal{U}_{\mathbf{U},T}$ to represent the set of all open-loop controls for the first player, and $\mathcal{V}_T$ for the set of all open-loop controls of the second player.

Remark 1.6. *There exist sufficient conditions that ensure the existence of a Nash equilibrium for the differential game (1.6) (e.g., Ivanov, 1997 and Williams, 1980).*

1.3 Sufficient Optimality Conditions

The following proposition provides sufficient conditions for the solution to the differential game (1.2) when the two players do not necessarily choose their optimal strategies simultaneously. This is, for example, the case with Stackelberg games, where one of the players has the advantage of choosing the optimal strategy first (the leader), and the other player (the follower) takes this strategy as given when minimizing the loss (see E. Dockner and Sorger, 2012). In some applications, Stackelberg games are a convenient representation of situations when, rather than facing an intelligent opponent, the minimizing agent plays against malevolent nature that acts as a leader and chooses disturbances such as to maximize the follower's loss. Solutions of such games result in optimal strategies that are robust to general uncertainty.

Proposition 1.7 (Stackelberg Equilibrium)**.** *Let the Standing Assumption (*) hold true, and let $(\hat{\mathbf{u}}, \hat{\mathbf{v}}) \in \mathcal{U}_{\mathbf{U},T} \times \mathcal{V}_T$ be a solution of the finite-time horizon differential game (1.6) , i.e.,*

$$J_T(x_0, t_0, \hat{\mathbf{u}}, \hat{\mathbf{v}}) = \inf_{\mathbf{u}\in\mathcal{U}_{\mathbf{U}}} \sup_{\mathbf{v}\in\mathcal{V}} J_T(x_0, t_0, \mathbf{u}, \mathbf{v}).$$

We assume that $\|x_{\hat{\mathbf{u}},\hat{\mathbf{v}}}(T)\| \leq \delta$ *(where* δ *is introduced in Proposition 1.3) and define* $(\hat{\mathbf{u}}_\infty, \hat{\mathbf{v}}_\infty)$ *as follows:*

$$(\hat{\mathbf{u}}_\infty, \hat{\mathbf{v}}_\infty) := \begin{cases} (\hat{\mathbf{u}}, \hat{\mathbf{v}}) & \text{on the interval } [t_0, T], \\ (\bar{\mathbf{u}}, \bar{\mathbf{v}}) & \text{on the interval } (T, +\infty) \end{cases}$$

($\bar{\mathbf{u}}$ *and* $\bar{\mathbf{v}}$ *are the closed-loop controls defined by (1.4)). Then,* $(\hat{\mathbf{u}}_\infty, \hat{\mathbf{v}}_\infty)$ *solves the infinite-horizon differential game (1.2), i.e.,*

$$J(x_0, t_0, \hat{\mathbf{u}}_\infty, \hat{\mathbf{v}}_\infty) = \inf_{\mathbf{u} \in \mathcal{U}_{\mathbf{U}}} \sup_{\mathbf{v} \in \mathcal{V}} J(x_0, t_0, \mathbf{u}, \mathbf{v}).$$

Remark 1.8. *An intriguing question arises at this point: Suppose* $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ *is a solution to the finite-time horizon differential game under the assumptions of Proposition 1.7. What would be the nature of* $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ *over the interval* $[0, T]$*? Our observation suggests that there exists a moment in time* $\hat{T} \in [0, T]$*, such that* $\hat{u}(t)$ *belongs to the boundary of the set* $\mathbf{U}$ *for each* $t \in [0, \hat{T}]$*. However, determining this explicitly remains an open question for us.*

If the finite-time horizon differential game (1.6) possesses a saddle point (Nash equilibrium), then the corollary below allows us to extend this equilibrium to the infinite-time horizon differential game (1.2).

Corollary 1.9 (Nash equilibrium). *Let the Standing Assumption (*) hold true, and let* $(\hat{\mathbf{u}}, \hat{\mathbf{v}}) \in \mathcal{U}_{\mathbf{U},T} \times \mathcal{V}_T$ *be a saddle point for the finite-horizon differential game (1.6), i.e.,*

$$J_T(x_0, t_0, \hat{\mathbf{u}}, \mathbf{v}) \leq J_T(x_0, t_0, \hat{\mathbf{u}}, \hat{\mathbf{v}}) \leq J_T(x_0, t_0, \mathbf{u}, \hat{\mathbf{v}})$$

for all $\mathbf{u} \in \mathcal{U}_{\mathbf{U},T}$ *and* $\mathbf{v} \in \mathcal{V}_T$*. We assume that* $\|x_{\hat{\mathbf{u}},\hat{\mathbf{v}}}(T)\| \leq \delta$ *(where* δ *is introduced in Proposition 1.3) and define* $(\hat{\mathbf{u}}_\infty, \hat{\mathbf{v}}_\infty)$ *as follows:*

$$(\hat{\mathbf{u}}_\infty, \hat{\mathbf{v}}_\infty) := \begin{cases} (\hat{\mathbf{u}}, \hat{\mathbf{v}}) & \text{on the interval } [t_0, T], \\ (\bar{\mathbf{u}}, \bar{\mathbf{v}}) & \text{on the interval } (T, +\infty) \end{cases}$$

($\bar{\mathbf{u}}$ *and* $\bar{\mathbf{v}}$ *are the closed-loop controls defined by (1.4)). Then,* $(\hat{\mathbf{u}}_\infty, \hat{\mathbf{v}}_\infty)$ *provides a saddle point (Nash equilibrium) for the infinite-horizon differential game (1.2), i.e.,*

$$J(x_0, t_0, \hat{\mathbf{u}}_\infty, \mathbf{v}) \leq J(x_0, t_0, \hat{\mathbf{u}}_\infty, \hat{\mathbf{v}}_\infty) \leq J(x_0, t_0, \mathbf{u}, \hat{\mathbf{v}}_\infty)$$

for each $\mathbf{u} \in \mathcal{U}_{\mathbf{U}}$ *and each* $\mathbf{v} \in \mathcal{V}$*.*

Remark 1.10. *Corollary 1.9 in fact enables us to find the solution of the infinite-time horizon differential game by transforming it into an equivalent finite-time horizon game, which can be addressed using appropriate numerical methods.*

Chapter 2

Discrete-Time Linear-Quadratic Game

"Change is the only constant in life."

Heraclitus (c. 535 – c. 475 BC)

2.1 Formulation of the Problem

Let k be an arbitrary non-negative integer. We denote by $\mathbb{N}_k$ the set of all non-negative integers greater than or equal to k. Let us fix an arbitrary non-negative integer $k_0 \in \mathbb{N}_0$ and a vector $x_0 \in \mathbb{R}^n$. We consider a class of discrete-time non-cooperative linear-quadratic games. The actions (controls) of the players are determined by their choice of functions: $\mathbf{u}$ for the first player and $\mathbf{v}$ for the second player.

For each positive integer m, we define the set

$$\ell^2(\mathbb{N}_{k_0}, \mathbb{R}^m) := \left\{ \left(w_{k_0}, w_{k_0+1}, w_{k_0+2}, \ldots\right) : \sum_{k=k_0}^{\infty} \|w_k\|^2 < \infty \right\},$$

which constitutes a Banach space with the norm $\|\mathbf{w}\|_2 = \left(\sum_{k=k_0}^{\infty} \|w_k\|^2\right)^{1/2}$.

Both players may use open-loop or closed-loop controls. An open-loop control $\mathbf{u} := \{u_k\}_{k \in \mathbb{N}_{k_0}}$ of the first player is a sequence belonging to $\ell^2(\mathbb{N}_{k_0}, \mathbb{R}^{m_u})$ and satisfying the relations $u_k \in U_k \subset \mathbb{R}^{m_u}$ for each $k \in \mathbb{N}_{k_0}$. We denote by $\mathcal{U}_{\mathrm{U}}$ the set of all open-loop controls of the first player

$$\mathcal{U}_{\mathrm{U}} := \left\{ \mathbf{u} = (u_{k_0}, u_{k_0+1}, \ldots, u_{k_0+k}, \ldots) : u_i \in U_i, i = k_0, k_0+1, \ldots, k_0+k, \ldots \right\},$$

where

$$\mathbf{U} := U_{k_0} \times U_{k_0+1} \times ... \times U_{k_0+k} \times ....$$

In what follows, we will occasionally need the finite-time version of the game. Thus, for some $\kappa \in \mathbb{N}_{k_0}$, we will denote by $\mathcal{U}_{\mathbf{U}}^{\kappa}$ the set of admissible controls starting at time k_0 and ending at time κ, that is

$$\mathcal{U}_{\mathbf{U}}^{\kappa} := \left\{ \mathbf{u}^{\kappa} = (u_{k_0}, u_{k_0+1}, ..., u_{\kappa}) : \ u_i \in U_i, i = k_0, k_0+1, ..., \kappa \right\}. \tag{2.1}$$

When we consider a shorter time interval, i.e., $k > k_0$, it will be indicated by a subscript in the notation as follows: $\mathcal{U}_{k,\mathbf{U}}^{\kappa}$. This notational convention will also be applied in the case where there are no constraints imposed on the player's controls ($U_k = \mathbb{R}^{m_u}$), except that in this case, the subscript $\mathbf{U}$ will be omitted for simplicity. So, for instance, we will write $\mathcal{U}$ for the set of admissible (unconstrained) controls on the infinite-time horizon starting from k_0, and $\mathcal{U}_k^{\kappa}$ for the set of admissible controls starting at k and ending at κ.

Throughout the chapter, we assume that there are no control constraints for the second player. An open-loop control $\mathbf{v} := \{v_k\}_{k \in \mathcal{N}_{k_0}}$ of the second player is a sequence, belonging to $\ell^2(\mathbb{N}_{k_0}, \mathbb{R}^{m_v})$ and satisfying the relations $v_k \in \mathbb{R}^{m_v}$ for each $k \in \mathbb{N}_{k_0}$. We denote by $\mathcal{V}$ the set of all open-loop controls of the second player

$$\mathcal{V} := \left\{ \mathbf{v} = (v_{k_0}, v_{k_0+1}, ..., v_{k_0+k}, ...) : \ v_i \in \mathbb{R}^{m_v}, i = k_0, k_0+1, ..., k_0+\kappa, ... \right\}.$$

As previously, for each $\kappa \in \mathbb{N}_{k_0}$, we will denote by $\mathcal{V}^{\kappa}$ the finite-dimensional projection of $\mathcal{V}$

$$\mathcal{V}^{\kappa} := \left\{ \mathbf{v}^{\kappa} = (v_{k_0}, v_{k_0+1}, ..., v_k) : \ v_i \in \mathbb{R}^{m_v}, i = k_0, k_0+1, ..., \kappa \right\}. \tag{2.2}$$

The sets $\mathcal{V}_k$ and $\mathcal{V}_k^{\kappa}$ are defined in the same way as the corresponding control sets of the first player.

Let $\mathbf{u} \in \mathcal{U}_{\mathbf{U}}$ and $\mathbf{v} \in \mathcal{V}$ be admissible controls of the first and second players, respectively. The state trajectory

$$\mathbf{x} = (x_{k_0}, x_{k_0+1}, x_{k_0+2}, ...),$$

corresponding to this control pair is determined recursively by the dynamics of the game, which is described as follows:

$$x_{k+1} = Ax_k + B_u u_k + B_v v_k, \ \ x_{k_0} = x_0, \ \ k = k_0, k_0+1, k_0+2, ..., \tag{2.3}$$

where A, B_u, and B_v are matrices of dimensions $n \times n$, $n \times m_u$, and $n \times m_v$, respectively. Here x_k, $k \in \mathbb{N}_{k_0}$, and x_0 denote the state of the system at the moment of time k and the initial state at time k_0, respectively.

A closed-loop control

$$\mathbf{u}^c = (u_{k_0}^c, u_{k_0+1}^c, u_{k_0+2}^c, ...)$$

of the first player is a function of the form $\mathbf{x} \to \mathbf{u}^c(\mathbf{x}) \in \mathbf{U}$, where

$$\mathbf{u}^c(\mathbf{x}) := (u_{k_0}^c(x_{k_0}), u_{k_0+1}^c(x_{k_0+1}), ..., u_k^c(x_k), ...)$$

with $u_k^c(x_k) \in U_k$ for each $k = k_0, k_0+1, k_0+2, \ldots$. A closed-loop control

$$\mathbf{v}^c = (v_{k_0}^c, v_{k_0+1}^c, v_{k_0+2}^c, \ldots)$$

of the second player is defined analogously. Note that the closed-loop controls depend only on the current state rather than the entire history of the process (Markov property).

Let $\mathbf{u}^c$ and $\mathbf{v}^c$ be arbitrary admissible closed-loop controls of the first and second players, respectively. The state trajectory

$$\mathbf{x} = (x_{k_0}, x_{k_0+1}, x_{k_0+2}, \ldots),$$

corresponding to this control pair is determined recursively as follows:

$$x_{k+1} = Ax_k + B_u u_k^c(x_k) + B_v v_k^c(x_k), \quad x_{k_0} = x_0, \quad k = k_0, k_0+1, k_0+2, \ldots .$$

If the function $\mathbf{u}^c$ is linear, then $u_k^c(x_k) = K_{u_k} x_k$, where each K_{u_k}, $k \in \mathbb{N}_{k_0}$, is a matrix of dimension $m_u \times n$. Analogously, if $\mathbf{v}^c$ is linear, then $v_k^c(x_k) = K_{v_k} x_k$, where each K_{v_k}, $k \in \mathbb{N}_{k_0}$, is a matrix of dimension $m_v \times n$.

It is clear that one can define similarly admissible controls of mixed type (part of their components are open-loop controls and the rest are closed-loop functions depending on the current state of the system). The corresponding trajectory is defined in a similar manner. An admissible control pair $(\mathbf{u}, \mathbf{v})$ is considered **feasible** if, in addition, the criterion J as specified in (2.4) for this pair yields a finite value.

Across all admissible control pairs, we consider the following discrete-time infinite horizon linear-quadratic game:

$$\inf_{\mathbf{u} \in \mathcal{U}_{\mathrm{U}}} \sup_{\mathbf{v} \in \mathcal{V}} J(x_0, k_0, \mathbf{u}, \mathbf{v}), \tag{2.4}$$

where the objective function is defined as

$$J(x_0, k_0, \mathbf{u}, \mathbf{v}) := \sum_{k=k_0}^{\infty} \left(x_k^\intercal Q x_k + u_k^\intercal R_u u_k - v_k^\intercal R_v v_k \right) .$$

Here, Q is an $n \times n$ symmetric positive semi-definite matrix, and R_u and R_v are symmetric, positive definite matrices of dimensions $m_u \times m_u$ and $m_v \times m_v$, respectively. The objective is to find a pair of feasible controls $(\mathbf{u}, \mathbf{v})$ that solve the problem 2.4.

Concerning the information pattern, we assume that both players know all parameters of the dynamics (the matrices A, B_u, and B_v) and of the objective function (the matrices Q, R_u, and R_v). Moreover, we assume that both players have access to the current system state x_k, $k \in \mathbb{N}_{k_0}$.

Let k be a non-negative integer satisfying $k \geq k_0$, and let $x \in \mathbb{R}^n$. The **value function** $V_{\mathrm{U}} : \mathbb{R}^n \times \mathbb{N}_{k_0} \to \mathbb{R}$ of the game (2.3) $\div$ (2.4) is defined by

$$V_{\mathbf{U}}(x,k) := \inf_{\mathbf{u}\in\mathcal{U}_{k,\mathbf{U}}} \sup_{\mathbf{v}\in\mathcal{V}_k} J(x,k,\mathbf{u},\mathbf{v}).$$

In particular for $\mathbf{U} = \mathbb{R}^{\mathbf{m_u}} \times \mathbb{R}^{\mathbf{m_u}} \times \cdots \times \mathbb{R}^{\mathbf{m_u}} \times \cdots$, we omit the subscript $'\mathbf{U}'$, so

$$V(x,k) := \inf_{\mathbf{u}\in\mathcal{U}_k} \sup_{\mathbf{v}\in\mathcal{V}_k} J(x,k,\mathbf{u},\mathbf{v}).$$

2.2 Preliminaries

Classical dynamic optimization problems are often solved by using Bellman's principle of optimality. Recall that, according to it, an optimal policy has the property that, whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. This principle provides a recursive expression for the optimal value function, expressing the optimal cost-to-go in terms of the immediate cost and the optimal cost-to-go at the next time step. It serves as a foundation for developing algorithms for solving optimal control problems through dynamic programming methods.

Below, we show that this principle applies also to the discrete-time min-max optimal control problem and, hence, to the considered infinite-time horizon linear-quadratic game. Consider the more general version of the game: Given the dynamics

$$x_{k+1} = f_k(x_k, u_k, v_k), \;\; x_{k_0} = x_0, \;\; k = k_0, k_0+1, k_0+2, ...,$$

find a pair of admissible controls $(\mathbf{u},\mathbf{v})$ that solve

$$\inf_{\mathbf{u}\in\mathcal{U}_{\mathbf{U}}} \sup_{\mathbf{v}\in\mathcal{V}} J_g(x_0, k_0, \mathbf{u}, \mathbf{v}),$$

where the objective and the corresponding value function are defined, respectively, as

$$J_g(x_0,k_0,\mathbf{u},\mathbf{v}) := \sum_{k=k_0}^{\infty} g_k\left(x_k,u_k,v_k\right)$$

$$V^g_{\mathbf{U}}(x_0,k_0) := \inf_{\mathbf{u}\in\mathcal{U}_{\mathbf{U}}} \sup_{\mathbf{v}\in\mathcal{V}} J_g(x_0,k_0,\mathbf{u},\mathbf{v}).$$

Theorem 2.1. *(Bellman's principle of optimality) Let $k \geq k_0$, $x \in \mathbb{R}^n$, and let the value $V^g_{\mathbf{U}}(x,k)$ be a real number. Then the following equality is satisfied:*

$$V^g_{\mathbf{U}}(x,k) = \inf_{u_k\in U_k} \sup_{v_k\in\mathbb{R}^{m_v}} \left\{ g_k\left(x_k,u_k,v_k\right) + V^g_{\mathbf{U}}\left(f_k\left(x_k,u_k,v_k\right), k+1\right)\right\}.$$

Remark 2.2. *The same result can be easily derived to the more general case when constraints are imposed on the control of the second player. Moreover, the result holds if the supremum and infimum are interchanged. Thus, if we choose a non-negative integer*

$k \geq k_0$, a point $x \in \mathbb{R}^n$, and set

$$\overline{V}^g_{\mathbf{U}}(x,k) := \sup_{\mathbf{v}\in\mathcal{V}_k} \inf_{\mathbf{u}\in\mathcal{U}_{k,\mathbf{U}}} J_g(x,k,\mathbf{u},\mathbf{v}),$$

if $\overline{V}^g_{\mathbf{U}}(x,k)$ is a real number, then Bellman's principle of optimality implies that

$$\overline{V}^g_{\mathbf{U}}(x,k) = \sup_{v_k\in\mathbb{R}^{m_v}} \inf_{u_k\in U_k} \left\{ g_k(x_k,u_k,v_k) + \overline{V}^g_{\mathbf{U}}\left(f_k\left(x_k,u_k,v_k\right),k+1\right)\right\}.$$

The subsequent assumption, in force throughout the remainder of this chapter, defines an algebraic Riccati equation (2.5) for the game and specifies particular conditions for a matrix P that solves this equation. Broadly, these conditions pertain to the existence of solutions of the maximization and minimization problems and guarantee properties like symmetry, positive definiteness, and specific inequalities for P and related matrices in the Riccati equation. Moreover, the requirements play a critical role in the stability and performance analysis of the considered control systems.

StandingAssumption : (**)

The matrix P is a symmetric positive definite solution of the following algebraic Riccati equation:

$$P = Q + A^{\intercal}(L^{-1})^{\intercal}PA, \tag{2.5}$$

where

$$L := I + (B_uR_u^{-1}B_u^{\intercal} - B_vR_v^{-1}B_v^{\intercal})P$$

is an invertible matrix. In addition, we assume that $\|A\| < 1$, and the matrices

$$[P - \tilde{A}^{\intercal}P\tilde{A}] \quad \text{and} \quad [R_v - B_v^{\intercal}PB_v]$$

are positive definite, where

$$\tilde{A} := A - (B_uR_u^{-1}B_u^{\intercal} - B_vR_v^{-1}B_v^{\intercal})PL^{-1}A.$$

Here, the norm of the matrix is defined as:

$$\|A\| = \max_{\|x\|=1} \|Ax\|.$$

The proposition we introduce next provides a min-max type result for the unconstrained game. This equivalence holds significant importance in game theory and optimization, especially in scenarios characterized by competitive or adversarial relationships. While the existing literature is replete with min-max theorems, these often stipulate the compactness of at least one of the participating sets. In our study, however, we demonstrate that for a quadratic objective function, such compactness conditions are not necessary.

Proposition 2.3. *Let the Standing Assumption (**) hold true. Then*

$$\min_{u\in\mathbb{R}^{m_u}} \max_{v\in\mathbb{R}^{m_v}} \{x^\top Qx + u^\top R_u u - v^\top R_v v + (Ax + B_u u + B_v v)^\top P(Ax + B_u u + B_v v)\}$$

$$= \max_{v\in\mathbb{R}^{m_v}} \min_{u\in\mathbb{R}^{m_u}} \{x^\top Qx + u^\top R_u u - v^\top R_v v + (Ax + B_u u + B_v v)^\top P(Ax + B_u u + B_v v)\}$$

for each point $x \in \mathbb{R}^n$.

2.3 Approximation of the Constrained Game

As noted earlier, the main idea of our approach is to approximate the game (2.3) ÷ (2.4) by some game on a finite-time horizon. Let $\mathbf{u} \in \mathcal{U}_\mathbf{U}$ and $\mathbf{v} \in \mathcal{V}$ be arbitrary admissible controls of the first and second players, respectively. For each $\kappa \in \mathcal{N}_{k_0}$, we consider the game

$$\inf_{\mathbf{u}^\kappa\in\mathcal{U}_\mathbf{U}^\kappa} \sup_{\mathbf{v}^\kappa\in\mathcal{V}^\kappa} J^\kappa(x_0, k_0, \mathbf{u}^\kappa, \mathbf{v}^\kappa) \tag{2.6}$$

subject to the dynamics (2.3), with $\mathbf{u}^\kappa$ and $\mathbf{v}^\kappa$ being defined by (2.1) and (2.2), respectively. The game's objective function is defined in the following manner:

$$J^\kappa(x_0, k_0, \mathbf{u}^\kappa, \mathbf{v}^\kappa) := x_{\kappa+1}^\top P x_{\kappa+1} + \sum_{k=k_0}^{\kappa} \left(x_k^\top Q x_k + u_k^\top R_u u_k - v_k^\top R_v v_k\right),$$

where P is the symmetric positive definite solution of the algebraic Riccati equation (2.5). The value function $V_\mathbf{U}^\kappa : \mathbb{R}^n \times \{k_0, k_0+1, k_0+2, ..., \kappa\} \to \mathbb{R}$ of this game is defined as

$$V_\mathbf{U}^\kappa(x_0, k_0) := \inf_{\mathbf{u}^\kappa\in\mathcal{U}_\mathbf{U}^\kappa} \sup_{\mathbf{v}^\kappa\in\mathcal{V}^\kappa} J^\kappa(x_0, k_0, \mathbf{u}^\kappa, \mathbf{v}^\kappa).$$

Let us fix an arbitrary point $x \in \mathbb{R}^n$, and let k be an arbitrary element of the index set

$$\mathcal{I}_{k_0}^\kappa := \{k_0, k_0+1, ..., \kappa\}.$$

Then, clearly,

$$V_\mathbf{U}^\kappa(x, \kappa) = x^\top P x. \tag{2.7}$$

Also, we obtain from Bellman's principle of optimality

$$V_\mathbf{U}^\kappa(x, k) = \inf_{u_k\in U_k} \sup_{v_k\in\mathbb{R}^{m_v}} \{x^\top Qx + u_k^\top R_u u_k - v_k^\top R_v v_k + V_\mathbf{U}^\kappa(Ax + B_u u_k + B_v v_k, k+1)\}.$$

Note that (2.7) combined with the equality above allows us to explicitly calculate the value function $V_\mathbf{U}^\kappa(x, k)$ not only for the matrix P but for an arbitrary positive definite matrix. Suppose that we know the value function $V_\mathbf{U}^\kappa(x, k)$.
Next, we examine the asymptotic properties of the system's trajectory. This is essential to establish the link between the game on an infinite-time horizon (2.6) and a suitable finite-horizon game (2.4).

Proposition 2.4. *Let the Standing Assumption (**) hold true, let $k_0 \in \mathbb{N}_0$, let*

$$(\mathbf{u}, \mathbf{v}) \in \ell^2(\mathbb{N}_{k_0}, \mathbb{R}^{m_u}) \times \ell^2(\mathbb{N}_{k_0}, \mathbb{R}^{m_v})$$

be arbitrary ℓ^2-sequences, and let $\mathbf{x} = (x_{k_0}, x_{k_0+1}, x_{k_0+2}, ...)$ be the corresponding state trajectory. Then

$$\lim_{k\uparrow+\infty} x_k = \mathbf{0}.$$

The following theorem establishes a condition for the existence of a Nash equilibrium for the unconstrained game (2.3) ÷ (2.4). After deriving the solution of the unconstrained game, which takes the form of linear closed-loop controls, we will show how it can be used to solve the original problem where the controls of the minimizing player are subject to constraints. In particular, we will prove that there exists a moment of time when the constraints become no longer binding, and from that moment on, the unconstrained optimal controls are applied.

Theorem 2.5. *Let the Standing Assumption (**) hold true. We define matrices K_u and K_v as follows:*

$$K_u = R_u^{-1} B_u^\intercal P L^{-1} A \quad \text{and} \quad K_v = R_v^{-1} B_v^\intercal P L^{-1} A, \tag{2.8}$$

and assume that $\|A - B_u K_u\| < 1$, $\|A + B_v K_v\| < 1$ and $\|A - B_u K_u + B_v K_v\| < 1$. Then

$$\begin{aligned} V(x_0, k_0) &= x_0^\intercal P x_0 \\ &= J(x_0, k_0, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \\ &= \min_{\mathbf{u}\in\mathcal{U}} \max_{\mathbf{v}\in\mathcal{V}} J(x_0, k_0, \mathbf{u}, \mathbf{v}) \\ &= \max_{\mathbf{v}\in\mathcal{V}} \min_{\mathbf{u}\in\mathcal{U}} J(x_0, k_0, \mathbf{u}, \mathbf{v}), \end{aligned} \tag{2.9}$$

where

$$\tilde{\mathbf{u}} := (\tilde{u}_{k_0}, \tilde{u}_{k_0+1}, \tilde{u}_{k_0+2}, ...) \text{ and } \tilde{\mathbf{v}} := (\tilde{v}_{k_0}, \tilde{v}_{k_0+1}, \tilde{v}_{k_0+2}, ...)$$

are linear closed-loop controls defined as $\tilde{u}_k(x) = -K_u x$ and $\tilde{v}_k(x) = K_v x$ for each $k \in \mathbb{N}_{k_0}$ and for every point $x \in \mathbb{R}^n$.

We now return to the original game, where constraints on the control of the first player are present. To establish the next result, we need an additional assumption about the control sets of the minimizing player, namely that the sets $U_k, k \in \mathbb{N}_{k_0}$ contain a convex neighborhood U of the origin in $\mathbb{R}^{m_u}$. Under this assumption, the next proposition establishes the existence of a neighborhood of the origin in $\mathbb{R}^n$, where these control constraints are no longer binding.

Let us denote by $\tilde{\mathbf{x}}_\kappa^y := (\tilde{x}_\kappa^y, \tilde{x}_{\kappa+1}^y, ...)$ the trajectory of the discrete system

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k, \quad \tilde{x}_\kappa = y, \; k \in \mathbb{N}_\kappa$$

(starting from the point y at the moment τ), where the matrix $\tilde{A}$ is introduced in the Standing Assumption (**), i.e., $\tilde{\mathbf{x}}_\kappa^y$ is induced by the closed-loop controls $-K_u x$ and $K_v x$ defined by (2.8).

Proposition 2.6. *Let all sets $U_k, k = k_0, k_0 + 1, ...,$ contain a convex neighborhood U of the origin in $\mathbb{R}^{m_u}$, and let the Standing Assumption (**) and the assumptions of Theorem 2.5 be satisfied. Then there exist a positive real number δ_0 and another real number $\delta \in (0, \delta_0)$ such that:*

i) $-K_u x \in U$ for each $x \in \delta_0 \bar{\mathbf{B}}^n$, where $\bar{\mathbf{B}}^n$ is the closed unit ball in $\mathbb{R}^n$;

ii) for every point $y \in \delta \bar{\mathbf{B}}^n$ and for every $\kappa \geq k_0$, the inclusion $\tilde{x}_k^y \in \delta_0 \bar{\mathbf{B}}^n$ holds true for all $k \in \mathbb{N}_\kappa$ and the sequence $\{\tilde{x}_k^y\}_{k=\kappa}^{\infty}$ tends to the origin as $k \to +\infty$.

Going forward, we will fix the parameter δ, introduced in Proposition 2.6.

Remark 2.7. *Notice that Proposition 2.4 implies the existence of a moment in time $\tau \geq k_0$ such that the state trajectory $\mathbf{x} = (x_{k_0}, x_{k_0+1}, x_{k_0+2}, \ldots)$ induced by an arbitrary pair of ℓ^2-sequences $(\mathbf{u}, \mathbf{v})$, starting from an arbitrary point $y \in \mathbb{R}^n$, enters the δ-neighborhood $\delta \bar{\mathbf{B}}^n$ of the origin, i.e., $x_\tau \in \delta \bar{\mathbf{B}}^n$.*

2.4 Sufficient Optimality Conditions

The ensuing theorem shows that, when constraints are imposed on the controls of the first player, the value function of the infinite-time horizon game aligns with its finite-time horizon counterpart, provided an appropriate choice of the terminal moment κ is made.

Theorem 2.8. *Let the Standing Assumption (**) hold true, let $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ be an admissible control pair for the problem (2.3) $\div$ (2.4), $\delta > 0$ be the real number introduced in Proposition 2.6 and let $\bar{\mathbf{x}}$ be the corresponding state trajectory. If the equality*

$$V_{\mathbf{U}}(x_0, k_0) = J(x_0, k_0, \bar{\mathbf{u}}, \bar{\mathbf{v}})$$

holds then, there exists a non-negative integer $\kappa \geq k_0$ such that $\bar{x}_\kappa \in \delta \bar{\mathbf{B}}$ and

$$V_{\mathbf{U}}(x_0, k_0) = J(x_0, k_0, \bar{\mathbf{u}}^\kappa, \bar{\mathbf{v}}^\kappa) = V_{\mathbf{U}}^\kappa(x_0, k_0),$$

where $\bar{\mathbf{u}}^\kappa$ and $\bar{\mathbf{v}}^\kappa$ are defined by (2.1) and (2.2).

Having established the equivalence between the infinite-time horizon game and a suitably defined game on a finite-time horizon, the next result demonstrates how the constrained game can be solved using the solution of the corresponding unconstrained one. The following corollary essentially asserts that if we have the solution to the game on a finite-time horizon and we extend it by applying the linear closed-loop controls of the unconstrained game, we would obtain the solution of the infinite-time horizon constrained game.

Corollary 2.9. *Let the Standing Assumption (**) hold true, let $(\bar{\mathbf{u}}, \bar{\mathbf{v}}) \in \mathcal{U}_{\mathbf{U}} \times \mathcal{V}$ be an admissible control pair for the problem (2.3) $\div$ (2.4), and let $\kappa \geq k_0$ be sufficiently large so that the point $\bar{x}_\kappa \in \bar{\mathbf{x}}^\kappa$ belongs to $\delta \bar{\mathbf{B}}^n$ (δ is introduced in Proposition 2.6), where $\bar{\mathbf{x}}^\kappa$ is the trajectory corresponding to the control pair $(\bar{\mathbf{u}}^\kappa, \bar{\mathbf{v}}^\kappa)$. Let*

$$V_{\mathbf{U}}^\kappa(x_0, k_0) = J^\kappa(x_0, k_0, \bar{\mathbf{u}}^\kappa, \bar{\mathbf{v}}^\kappa),$$

then

$$V_{\mathbf{U}}^{\kappa}(x_0, k_0) = J(x_0, k_0, \hat{\mathbf{u}}, \hat{\mathbf{v}}) = V_{\mathbf{U}}(x_0, k_0),$$

where

$$(\hat{\mathbf{u}}, \hat{\mathbf{v}}) := \begin{cases} (\bar{\mathbf{u}}^{\kappa}, \bar{\mathbf{v}}^{\kappa}) & \textit{on the set } \{k_0, k_0+1, ..., \kappa\}, \\ (\mathbf{u}_{\kappa+1}^{c}, \mathbf{v}_{\kappa+1}^{c}) & \textit{on the set } \{\kappa+1, \kappa+2, ...\}, \end{cases}$$

with $u_k^c(x_k) = -K_u x_k$ *and* $v_k^c(x_k) = K_v x_k$ *for each* $k = \kappa+1, \kappa+2, ...,$ *where the matrices* K_u *and* K_v *are defined in Theorem* 2.5.

Chapter 3

Discrete-Time Dynamic Game

"Nature does not hurry, yet everything is accomplished. "

Lao Tzu (6th century BC)

3.1 Formulation of the Discrete-Time Game

Denote by $\mathbb{N}$ the set of all non-negative integers and $\mathbb{R}^n$ the n-dimensional Euclidean space. We fix a vector x_0 in an open subset $G \subset \mathbb{R}^n$ and consider a two-person discrete-time dynamic game on an infinite time horizon. The dynamics of the game is described by the following discrete-time control system:

$$x_{k+1} = f_k(x_k, u_k, v_k), \quad x_{k_0} = x_0,\ u_k \in U_k,\ v_k \in V_k,\ k \in \mathbb{N}. \tag{3.1}$$

The notation and assumptions are further clarified as follows:

- $\mathbb{N}$ represents the set of all non-negative integers, indicating the time steps of the system.
- $x_0 \in \mathbb{R}^n$ is the initial state vector, located in an open subset $G \subset \mathbb{R}^n$.
- $k_0 \in \mathbb{N}$ is the initial time, assumed to be 0 for simplicity.
- $x_k \in \mathbb{R}^n$ denotes the state of the system at time k.
- $U_k \subseteq \mathbb{R}^{m_u}$ and $V_k \subseteq \mathbb{R}^{m_v}$, for each $k \in \mathbb{N}$, are non-empty, closed, and convex sets representing the control action spaces for the two players.
- The function $f_k : G \times \tilde{U}_k \times \tilde{V}_k \to \mathbb{R}^n$, for each $k \in \mathbb{N}$, is continuously differentiable.

- $\tilde{U}_k$ and $\tilde{V}_k$ are open sets that contain U_k and V_k respectively, for each $k \in \mathbb{N}$, expanding the domain of the function f_k to facilitate its continuous differentiability.

Both players influence the system through their choice of functions $u : \mathbb{N} \to \mathbb{R}^{m_u}$ for the first player and $v : \mathbb{N} \to \mathbb{R}^{m_v}$ for the second player.

We call a pair of control sequences $(\mathbf{u}, \mathbf{v})$, where $\mathbf{u} := \{u_k\}_{k=0}^{\infty}$, $\mathbf{v} := \{v_k\}_{k=0}^{\infty}$ **admissible** if for every $k \in \mathbb{N}$, the inclusions $u_k \in U_k$ and $v_k \in V_k$ hold true. By $\mathcal{U}$ and $\mathcal{V}$, we denote the sets of all admissible strategies of the first and second players, respectively.

For a given admissible control pair $(\mathbf{u}, \mathbf{v})$, the equality (3.1) generates the trajectory $x_0, x_1, \ldots$. Note that this trajectory may be extended either to infinity or to the minimal number k, for which the following relation holds true: $f_k(x_k, u_k, v_k) \notin G$ (if such a k exists). In the former case, we call the triple $(\mathbf{x}, \mathbf{u}, \mathbf{v})$, where $\mathbf{x} := \{x_k\}_{k=0}^{\infty}$ an **admissible process**.

Given an admissible process $(\mathbf{x}, \mathbf{u}, \mathbf{v})$, the state trajectory $\{x_k\}_{k=0}^{\infty}$ can be represented from (3.1) as

$$x_{k+1} := f_k^{(u_k,v_k)} \circ f_{k-1}^{(u_{k-1},v_{k-1})} \circ \ldots \circ f_0^{(u_0,v_0)}(x_0), \quad k \in \mathbb{N}.$$

The symbol "$\circ$" represents the composition of the corresponding maps, and $f_k^{(u,v)}(x)$ denotes $f_k(x, u, v)$.

Given the dynamics (3.1), we consider the following discrete-time infinite horizon dynamic game:

$$\min_{\mathbf{u} \in \mathcal{U}} J(x_0, \mathbf{u}, \mathbf{v}), \quad \max_{\mathbf{v} \in \mathcal{V}} J(x_0, \mathbf{u}, \mathbf{v}), \tag{3.2}$$

whose criterion is defined as:

$$J(x_0, \mathbf{u}, \mathbf{v}) = \sum_{k=0}^{\infty} g_k(x_k, u_k, v_k). \tag{3.3}$$

It is evident from (3.2) that the first player strives to "minimize" this criterion, while the aim of the second player is to "maximize" it. Here, the functions $g_k : G \times \tilde{U}_k \times \tilde{V}_k \to \mathbb{R}$, $k \in \mathbb{N}$, are assumed to be continuously differentiable.

In infinite horizon optimal control problems, as the objective is to maximize or minimize some performance measure over an infinite time span the traditional optimization criteria, such as minimizing or maximizing an objective function over a finite time horizon, cannot be directly applied since the series (integrals) involved may not converge.

The weakly overtaking optimality criterion provides a solution to this issue. It states that a control is weakly overtaking optimal if, for any given alternative control, there exists a time after which the cumulative cost (or reward) of the optimal control is always better than the cumulative cost (or reward) of the alternative control. This does not require the difference between the cumulative costs (or rewards) to grow indefinitely but only that the optimal control is not worse than any other control as time goes to infinity.

To clarify the meaning of this optimal control problem, for any $K \in \mathbb{N}$ we denote

$$J_K(x_0, \mathbf{u}, \mathbf{v}) = \sum_{k=0}^{K} g_k(x_k, u_k, v_k).$$

Definition 3.1. *An admissible process* $(\bar{\mathbf{x}}, \bar{\mathbf{u}}, \bar{\mathbf{v}})$ *is called* **weakly overtaking optimal** *if for each* $\varepsilon > 0$*, for every positive integer* K *and for each admissible process* $(\mathbf{x}, \mathbf{u}, \mathbf{v})$ *there exists a positive integer* $\kappa > K$ *such that*

$$J_\kappa(x_0, \bar{\mathbf{u}}, \mathbf{v}) - \varepsilon \leq J_\kappa(x_0, \bar{\mathbf{u}}, \bar{\mathbf{v}}) \leq J_\kappa(x_0, \mathbf{u}, \bar{\mathbf{v}}) + \varepsilon.$$

The pair $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$, which forms a weakly overtaking optimal process, is referred to as a **weakly overtaking Nash equilibrium**.

Let $(\bar{\mathbf{x}}, \bar{\mathbf{u}}, \bar{\mathbf{v}})$ be a weakly overtaking optimal process. For every $k \in \mathbb{N}$ and for each vector ξ, we denote by $x^{k,\xi} = (x_k, x_{k+1}, \ldots)$ the trajectory induced by (3.1) with an initial state of $x_k = \xi$ at time k, i.e.,

$$x_{s+1}^{k,\xi} := f_s^{(\bar{u}_s, \bar{v}_s)} \circ f_{s-1}^{(\bar{u}_{s-1}, \bar{v}_{s-1})} \circ \ldots \circ f_k^{(\bar{u}_k, \bar{v}_k)}(\xi), \quad s = k, k+1, \ldots .$$

Clearly, $x^{k,\xi}$ may happen to be an infinite sequence or may terminate at the minimal $s > k$ such that $f_s(x_s^{k,\xi}, \bar{u}_s, \bar{v}_s) \notin G$.

Remark 3.2. *The above Definition 3.1, providing the meaning of the problem, is a modification of the one in Aseev, Krastanov, and Veliov, 2017 in the context of a dynamical game.*

In this line, similar to the cited above paper, we introduce the following assumption and definitions.

Assumption 3.3. *For every* $k \in \mathbb{N}$*, there exist a real number* $\alpha_k > 0$ *and a sequence* $\{\beta_s^k\}_{s=k}^{\infty}$ *with* $\sum_{s=k}^{\infty} \beta_s^k < \infty$*, such that* $\alpha_k \mathbf{B}(\bar{x}_k) \subset G$*, for every* $\xi \in \alpha_k \bar{\mathbf{B}}(\bar{x}_k)$ *the sequence* $x^{k,\xi}$ *is infinite, and*

$$\sup_{\xi \in \alpha_k \bar{\mathbf{B}}(\bar{x}_k)} \left\| \frac{\partial}{\partial \xi} g_s(x_s^{k,\xi}, \bar{u}_s, \bar{v}_s) \right\| \leq \beta_s^k,$$

where $\alpha_k \bar{\mathbf{B}}(\bar{x}_k)$ *is a closed ball in* $\mathbb{R}^n$ *centered at* $\bar{x}_k$ *with a radius* α_k*.*

Assumption 3.3 implies that the series

$$\sum_{s=k}^{\infty} \frac{\partial}{\partial \xi} g_s(x_s^{k,\xi}, \bar{u}_s, \bar{v}_s), \quad k = 1, 2, \ldots$$

is absolutely convergent, uniformly with respect to $\xi \in \alpha_k \bar{\mathbf{B}}(\bar{x}_k)$.

By the identity

$$g_s(x_s^{k,\xi}, \bar{u}_s, \bar{v}_s) = g_s(f_{s-1}^{(\bar{u}_{s-1}, \bar{v}_{s-1})} \circ f_{s-2}^{(\bar{u}_{s-2}, \bar{v}_{s-2})} \circ \ldots \circ f_k^{(\bar{u}_k, \bar{v}_k)}(\xi), \bar{u}_s, \bar{v}_s)$$

and the chain rule we have that

$$\frac{\partial}{\partial \xi} g_s(x_s^{k,\xi}, \bar{u}_s, \bar{v}_s) = \frac{\partial}{\partial x} g_s(x_s^{k,\xi}, \bar{u}_s, \bar{v}_s) \prod_{i=s-1}^{k} \frac{\partial}{\partial x} f_i(x_i^{k,\xi}, \bar{u}_i, \bar{v}_i), \tag{3.4}$$

where the following notation is used:

$$\prod_{i=s-1}^{k} A_i := \begin{cases} A_{s-1}A_{s-2}...A_k & \text{if } s > k \\ I & \text{if } s \le k. \end{cases}$$

Here, $A_i := \frac{\partial}{\partial x} f_i(x_i^{k,\xi}, \bar{u}_i, \bar{v}_i)$ and I denotes the identity matrix of dimension $n \times n$. Also, here we use the symbol $\prod$ instead of the usual symbol $\prod$ to indicate that the "increment" of the running index i is -1 (since $s > k$).

Following again Aseev, Krastanov, and Veliov, 2017, we define the **adjoint sequence** $\psi := \{\psi_k\}_{k=1}^{\infty}$ as:

$$\psi_k = \sum_{s=k}^{\infty} \frac{\partial}{\partial \xi} g_s(x_s^{k,\xi}, \bar{u}_s, \bar{v}_s)_{|\xi = \bar{x}_k}, \quad k = 1, 2, \dots .$$

Remark 3.4. *Assumption 3.3 actually implies that* $\|\psi_k\| < \infty$, $k = 1, 2, \dots$. *Furthermore, we obtain from (3.4) that*

$$\begin{aligned} \psi_k &= \sum_{s=k}^{\infty} \frac{\partial}{\partial \xi} g_s(x_s^{k,\xi}, \bar{u}_s, \bar{v}_s)_{|\xi = \bar{x}_k} \\ &\quad (\textit{taking into account that } x_s^{k,\bar{x}_k} = \bar{x}_s) \\ &= \sum_{s=k}^{\infty} \frac{\partial}{\partial x} g_s(\bar{x}_s, \bar{u}_s, \bar{v}_s) \prod_{t=s-1}^{k} \frac{\partial}{\partial x} f_t(\bar{x}_t, \bar{u}_t, \bar{v}_t). \end{aligned} \tag{3.5}$$

The second equality in (3.5) implies that the adjoint sequence, as defined, satisfies the **adjoint equation**

$$\psi_k = \psi_{k+1} \frac{\partial}{\partial x} f_k(\bar{x}_k, \bar{u}_k, \bar{v}_k) + \frac{\partial}{\partial x} g_k(\bar{x}_k, \bar{u}_k, \bar{v}_k), \quad k = 1, 2, \dots . \tag{3.6}$$

This is verified by the following chain of equalities:

$$\begin{aligned} \psi_k &= \sum_{s=k}^{\infty} \frac{\partial}{\partial x} g_s(\bar{x}_s, \bar{u}_s, \bar{v}_s) \prod_{i=s-1}^{k} \frac{\partial}{\partial x} f_i(\bar{x}_i, \bar{u}_i, \bar{v}_i) \\ &= \frac{\partial}{\partial x} g_k(\bar{x}_k, \bar{u}_k, \bar{v}_k) + \Big(\sum_{s=k+1}^{\infty} \frac{\partial}{\partial x} g_s(\bar{x}_s, \bar{u}_s, \bar{v}_s) \prod_{i=s-1}^{k+1} \frac{\partial}{\partial x} f_i(\bar{x}_i, \bar{u}_i, \bar{v}_i) \Big) \frac{\partial}{\partial x} f_k(\bar{x}_k, \bar{u}_k, \bar{v}_k) \\ &= \frac{\partial}{\partial x} g_k(\bar{x}_k, \bar{u}_k, \bar{v}_k) + \psi_{k+1} \frac{\partial}{\partial x} f_k(\bar{x}_k, \bar{u}_k, \bar{v}_k). \end{aligned} \tag{3.7}$$

3.2 Preliminaries

The formulation of the results presented below necessitates the introduction of the following definition:

Definition 3.5 (Bouligand tangent cone $T_S(\bar{y})$). *Consider a Banach space Y and a nonempty closed subset S of Y. Let $\bar{y}$ be an arbitrary point of S. The set $T_S(\bar{y})$, defined as the collection of all $w \in Y$ for which there exist a sequence of positive real numbers $\{t^\mu\}_{\mu=1}^\infty \downarrow 0$ and a sequence of points $\{w^\mu\}_{\mu=1}^\infty \subset Y$ converging to w, satisfying $\bar{y} + t^\mu w^\mu \in S$ for each $\mu = 1, 2, ...,$ is known as the* **Bouligand tangent cone** *to the closed subset S at the point $\bar{y} \in S$ (cf., for example, Aubin and Frankowska, 2009, Chapter 4.1).*

The theorem provided in Aseev, Krastanov, and Veliov, 2017 as Theorem 2.2 establishes a local maximum condition for the Hamiltonian function. This condition is satisfied for a certain solution $\psi = \{\psi_k\}_{k=1}^\infty$, of the adjoint equation (3.6). To identify the "correct" solution to this equation, additional conditions are needed, usually in the form of a transversality condition on ψ_k, at $k \to \infty$. The assertion relies on Assumption 3.13, which guarantees that the definition of ψ yields a finite vector. The theorem is formulated as follows:

Theorem 3.6 (Theorem 2.2 in Aseev, Krastanov, and Veliov, 2017). *Let Assumption 3.13 be fulfilled, the pair $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ be a weakly overtaking optimal process (in the context of the OCP in Aseev, Krastanov, and Veliov, 2017), and let the adjoint sequence ψ be defined by (3.13). Then, for every $k \in \mathbb{N}$, the following local maximum condition holds true:*

$$\Big(\frac{\partial}{\partial v} g_k(\bar{x}_k, \bar{v}_k) + \psi_{k+1} \frac{\partial}{\partial v} f_k(\bar{x}_k, \bar{v}_k)\Big) w \leq 0 \quad \textit{for every } w \in T_{V_k}(\bar{v}_k),$$

where $T_{V_k}(\bar{v}_k)$ is the Bouligand tangent cone introduced by Definition 3.5. ♢

The next statement is a corollary of the main result in Aseev, Krastanov, and Veliov, 2017, Theorem 3.6. For its formulation, we introduce the matrices $Z_k, k \in \mathbb{N}$ defined as

$$Z_k := \frac{\partial}{\partial x} f_{k-1}(\bar{x}_{k-1}, \bar{u}_{k-1}, \bar{v}_{k-1}) \frac{\partial}{\partial x} f_{k-2}(\bar{x}_{k-2}, \bar{u}_{k-2}, \bar{v}_{k-2}) ... \frac{\partial}{\partial x} f_0(\bar{x}_0, \bar{u}_0, \bar{v}_0). \quad (3.8)$$

Corollary 3.7. *Let Assumption 3.3 be fulfilled and the triple $(\bar{\mathbf{x}}, \bar{\mathbf{u}}, \bar{\mathbf{v}})$ be a weakly overtaking optimal process. Let the adjoint sequence $\psi = \{\psi_k\}_{k=1}^\infty$ be defined by (3.5). Then, for every $k \in \mathbb{N}$, the following* **local maximum condition** *holds true:*

$$\Big(\frac{\partial}{\partial v} g_k(\bar{x}_k, \bar{u}_k, \bar{v}_k) + \psi_{k+1} \frac{\partial}{\partial v} f_k(\bar{x}_k, \bar{u}_k, \bar{v}_k)\Big) w \leq 0 \quad \text{for every } w \in T_{V_k}(\bar{v}_k) \quad (3.9)$$

as well as the **transversality condition**

$$\lim_{k \to +\infty} \psi_k Z_k = 0.$$

♢

Remark 3.8. *The function $\mathcal{H}_k : G \times \mathbb{R}^{m_u} \times \mathbb{R}^{m_v} \times \mathbb{R}^n \to \mathbb{R}$, known as the* **Hamiltonian** *and incorporating the adjoint sequence ψ, is defined for each $k \in \mathbb{N}$ as follows:*

$$\mathcal{H}_k(x, u, v, \psi) := g_k(x, u, v) + \psi f_k(x, u, v).$$

Using this function, the relations (3.1), (3.6), and (3.9) can be expressed as follows, respectively:

$$\begin{aligned} \bar{x}_{k+1} &= \frac{\partial}{\partial \psi_{k+1}} \mathcal{H}_k(\bar{x}_k, \bar{u}_k, \bar{v}_k, \psi_{k+1}), \\ \psi_k &= \frac{\partial}{\partial x_k} \mathcal{H}_k(\bar{x}_k, \bar{u}_k, \bar{v}_k, \psi_{k+1}), \end{aligned} \tag{3.10}$$

and

$$\frac{\partial}{\partial v} \mathcal{H}_k(\bar{x}_k, \bar{u}_k, \bar{v}_k, \psi_{k+1}) w \leq 0 \quad \text{for every } w \in T_{V_k}(\bar{v}_k).$$

3.3 Necessary Optimality Condition

A distinctive feature of discrete-time optimal control problems, compared to their continuous-time counterparts, is the local nature of Pontryagin's maximum principles type for finite horizons. Specifically, in the absence of additional concavity-type conditions (when addressing an optimal maximization control problem), the maximum condition associated with the Hamiltonian only serves as a necessary condition for a local maximum. This principle is also applicable to problems on infinite horizons, as explored below.

Let the triple $(\bar{\mathbf{x}}, \bar{\mathbf{u}}, \bar{\mathbf{v}})$ be a weakly overtaking optimal process and let the adjoint sequence $\psi = \{\psi_k\}_{k=1}^{\infty}$ be defined by (3.5). For the main result in this chapter, we need the following assumption:

Assumption 3.9. *The following conditions hold true:*

(i) The function $g_k(\bar{x}_k, \cdot, \bar{v}_k) : U_k \to \mathbb{R}$ is convex. When $\psi^j_{k+1} > 0$ (the j-component of ψ_{k+1}), the j-component $f^j_k(\bar{x}_k, \cdot, \bar{v}_k) : U_k \to \mathbb{R}$ of the vector function $f_k(\bar{x}_k, \cdot, \bar{v}_k) : U_k \to \mathbb{R}^n$ is convex and when $\psi^j_{k+1} \leq 0$, it is concave.

(ii) The function $g_k(\bar{x}_k, \bar{u}_k, \cdot) : V_k \to \mathbb{R}$ is concave; When $\psi^j_{k+1} \geq 0$ (the j-component of ψ_{k+1}), the j-component $f^j_k(\bar{x}_k, \bar{u}_k, \cdot) : V_k \to \mathbb{R}$ of the vector function $f_k(\bar{x}_k, \bar{u}_k, \cdot) : V_k \to \mathbb{R}^n$ is concave and when $\psi^j_{k+1} < 0$, it is convex.

Remark 3.10. *Assumption 3.9 implies that for every $k \in \mathbb{N}$:*

- *the function $\mathcal{H}_k(\bar{x}_k, \cdot, \bar{v}_k, \psi_{k+1}) : U_k \to \mathbb{R}$ is convex;*
- *the function $\mathcal{H}_k(\bar{x}_k, \bar{u}_k, \cdot, \psi_{k+1}) : V_k \to \mathbb{R}$ is concave.*

Theorem 3.11 (Necessary Optimality Condition). *Let the Assumptions 3.3 and 3.9 hold true. Let the control pair $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ determines a weakly overtaking Nash equilibrium for the problem (3.1) ÷ (3.2) and let the adjoint sequence $\psi = \{\psi_k\}_{k=1}^{\infty}$ be defined by (3.5).*

Then, the adjoint sequence $\psi = \{\psi_k\}_{k=1}^{\infty}$ *solves the adjoint system* (3.10)*, and for every* $k \in \mathbb{N}$*, the following conditions are satisfied:*

(i) $\min_{u \in U_k} \mathcal{H}_k(\bar{x}_k, u, \bar{v}_k, \psi_{k+1}) = \mathcal{H}_k(\bar{x}_k, \bar{u}_k, \bar{v}_k, \psi_{k+1}) = \max_{v \in V_k} \mathcal{H}_k(\bar{x}_k, \bar{u}_k, v, \psi_{k+1})$;

(ii) transversality condition

$$\lim_{k \to +\infty} \psi_k Z_k = 0,$$

where Z_k *is defined by* (3.8)*.*

3.4 Discrete-Time Optimal Control Problem (OCP)

In the broad spectrum of dynamical games, optimal control problems without disturbances emerge as a distinct and specialized subset.

By contextualizing optimal control as a partial case of dynamical games, we recognize that the dynamics of a system responds to a single controller's decisions. In this perspective, optimal control can be seen as a singular player within the intricate web of dynamic game scenarios. The player's goal remains the same – to determine optimal control inputs.

The section establishes a new sufficient optimality condition under specific assumptions and shows its relevance to proving the minimum for the infinite-horizon optimal control problem. It introduces a series of conditions, derivations, leading to the theoretical foundations of optimality.

The subsequent sections will delve into the formulations of the problem.

3.4.1 Formulation of the Optimal Control Problem

In this scenario, the discrete-time control system is formulated to operate without external disturbances by setting their representative terms to a default value, chosen here as zero for simplicity. The system is defined as follows:

$$x_{k+1} = f_k(x_k, u_k), \quad x_{k_0} = x_0, \quad u_k \in U_k, \quad k \in \mathbb{N}. \tag{3.11}$$

The notation and assumptions are further clarified as follows:

- $\mathbb{N}$ denotes the set of all non-negative integers, including zero, indicating the time steps of the system.
- $k_0 \in \mathbb{N}$ represents the starting time, which is provisioned for simplicity to be 0.
- $x_0 \in \mathbb{R}^n$ specifies the initial state or point from which the system begins.
- Each U_k, for $k \in \mathbb{N}$, is a non-empty, closed, and convex subset of $\mathbb{R}^m$, defining the set of permissible control actions at each time step.
- The function $f_k : \mathbb{R}^n \times \tilde{U}_k \to \mathbb{R}^n$, for each $k \in \mathbb{N}$, is continuously differentiable. This function dictates the system's evolution from state x_k to x_{k+1} under the influence of control u_k.
- $\tilde{U}_k$ is an open subset in $\mathbb{R}^m$ that includes the set U_k, expanding the domain of the function f_k to facilitate its continuous differentiability.
- n and m are the dimensions of the respective vector spaces.

The function influencing the system, considered as a control sequence, is denoted by

$$\mathbf{u} := (u_0, u_1, ..., u_k, ...).$$

A control sequence is considered **admissible** if each of its components satisfies the following inclusion criteria: $u_k \in U_k$, $k \in \mathbb{N}$.

Assumption 3.12. *Every admissible control sequence* $\mathbf{u}$ *is an element of a Banach space* $(\mathbb{X}, \|\cdot\|)$.

For every given admissible control sequence $\mathbf{u}$, the equation (3.11) yields trajectory $\mathbf{x} = (x_0, x_1, ..., x_k, ...)$. The resulting pair $(\mathbf{x}, \mathbf{u})$ is referred to as an **admissible process**. For an admissible process $(\mathbf{x}, \mathbf{u})$, the state trajectory $\mathbf{x}$ is represented trough (3.11) as:

$$x_{k+1} := f_k^{(u_k)} \circ f_{k-1}^{(u_{k-1})} \circ \cdots \circ f_0^{(u_0)}(x_0), \quad k \in \mathbb{N}.$$

Here, the symbol $\circ$ denotes the composition of the corresponding maps, and $f_k^{(u)}(x)$ is defined as $f_k(x, u)$. An admissible process $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ is termed as **feasible** if $J(x_0, \bar{\mathbf{u}})$ is a real number, where the **objective function** (or simple the **objective**) is given by

$$J(x_0, \mathbf{u}) := \sum_{k=0}^{\infty} g_k(x_k, u_k).$$

In this setup, each function $g_k : \mathbb{R}^n \times \tilde{U}_k \to \mathbb{R}, \quad k \in \mathbb{N}$, is assumed to be continuously differentiable.

Over all admissible processes $(\mathbf{x}, \mathbf{u})$, we consider the following optimal control problem:

$$J(x_0, \mathbf{u}) \to \min . \tag{3.12}$$

We refer to an admissible control sequence $\mathbf{u}^*$ as **local optimal** if there exists a neighborhood N of $\mathbf{u}^*$ such that for any other admissible control sequence $\mathbf{u} \in N$, the inequality

$$J(x_0, \mathbf{u}^*) \leq J(x_0, \mathbf{u})$$

holds. If this inequality is satisfied by all admissible control sequences, then the control sequence $\mathbf{u}^*$ is referred to as the **global optimal**.

The optimal control sequence $\mathbf{u}^*$ leads to an optimal trajectory $\mathbf{x}^*$, which is the state trajectory generated by applying $\mathbf{u}^*$ to the system dynamics as defined in equation (3.11). The resulting pair $(\mathbf{x}^*, \mathbf{u}^*)$ is referred to as an **local (glabal) optimal process**.

Let $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ be any arbitrary feasible process. For each positive integer k, and every vector ξ, we represent by $x^{k,\xi} = (x_k, x_{k+1}, ...)$ the trajectory induced by (3.11) with the initial state $x_k = \xi$ at the time instant k, i.e.,

$$x_{s+1}^{k,\xi} := f_s^{(\bar{u}_s)} \circ f_{s-1}^{(\bar{u}_{s-1})} \circ \cdots \circ f_k^{(\bar{u}_k)}(\xi), \quad s = k, k+1, ...$$

As stated in Aseev, Krastanov, and Veliov, 2017, we adopt the following assumption, representing the disturbance-free variant of Assumption 3.3:

Assumption 3.13. *For every positive integer* k*, there exist a real number* $\alpha_k > 0$ *and a sequence* $\{\beta_s^k\}_{s=k}^{\infty}$ *with* $\sum_{s=k}^{\infty} \beta_s^k < \infty$ *such that the following inequality holds true:*

$$\sup_{\xi \in \alpha_k \bar{\mathbf{B}}(\bar{x}_k)} \left\| \frac{\partial}{\partial \xi} g_s(x_s^{k,\xi}, \bar{u}_s) \right\| \leq \beta_s^k, \quad s = k, k+1, ... ,$$

where $\alpha_k \bar{\mathbf{B}}(\bar{x}_k)$ is the closed ball in $\mathbb{R}^n$ centered at $\bar{x}_k$ with radius α_k.

Assumption 3.13 implies that, like its counterpart in the presence of disturbances, the sequence

$$\sum_{s=k}^{\infty} \frac{\partial}{\partial \xi} g_s(x_s^{k,\xi}, \bar{u}_s), \quad k = 1, 2, \ldots,$$

is absolutely convergent. Moreover this convergence is uniform with respect to all points ξ within the ball $\alpha_k \bar{\mathbf{B}}(\bar{x}_k)$.

Additionally, through the identity

$$g_s(x_s^{k,\xi}, \bar{u}_s) = g_s\big(f_{s-1}^{(\bar{u}_{s-1})} \circ f_{s-2}^{(\bar{u}_{s-2})} \circ \cdots \circ f_k^{(\bar{u}_k)}(\xi), \bar{u}_s\big)$$

the application of the chain rule enables us to deduce that

$$\frac{\partial}{\partial \xi} g_s(x_s^{k,\xi}, \bar{u}_s) = \frac{\partial}{\partial x} g_s(x_s^{k,\xi}, \bar{u}_s) \prod_{i=s-1}^{k} \frac{\partial}{\partial x} f_i(x_i^{k,\xi}, \bar{u}_i).$$

In accordance with Aseev, Krastanov, and Veliov, 2017, the no-disturbance variant of the **adjoint sequence** $\psi := \{\psi_k\}_{k=1}^{\infty}$ is defined as follows:

$$\psi_k = \sum_{s=k}^{\infty} \frac{\partial}{\partial \xi} g_s(x_s^{k,\xi}, \bar{u}_s)_{|\xi=\bar{x}_k} = \sum_{s=k}^{\infty} \frac{\partial}{\partial x} g_s(\bar{x}_s, \bar{u}_s) \prod_{i=s-1}^{k} \frac{\partial}{\partial x} f_i(\bar{x}_i, \bar{u}_i). \tag{3.13}$$

With the application of the second equality in (3.13), it becomes evident that the adjoint sequence defined in this manner satisfies the **adjoint equation**.

$$\psi_k = \frac{\partial}{\partial x} g_k(\bar{x}_k, \bar{u}_k) + \psi_{k+1} \frac{\partial}{\partial x} f_k(\bar{x}_k, \bar{u}_k), \quad k = 1, 2, \ldots,$$

as shown in (3.7).

Using ψ, for every $k \in \mathbb{N}$, we introduce the corresponding **Hamiltonian** function $\mathcal{H}_k : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\mathcal{H}_k(x, u, \psi) := g_k(x, u) + \psi f_k(x, u).$$

3.4.2 The Objective-Hamiltonian Relation

In order to present the results derived in this section, we adopt the definition as outlined by Aubin, 1984.

Definition 3.14 (Derivative in direction of $T_S(\bar{y})$)**.** *Let Y be a Banach space, S be a nonempty closed subset of Y, $\bar{y}$ be an arbitrary point of S, and w be an arbitrary element of the Bouligand tangent cone $T_S(\bar{y})$, introduced by Definition 3.5. We say that the function $h : S \to \mathbb{R}$ is differentiable in direction w if the following limit exists:*

$$\lim_{t^\mu \downarrow 0} \frac{h(\bar{y} + t^\mu w^\mu) - h(\bar{y})}{t^\mu},$$

where the sequence $\{t^\mu\}_{\mu=0}^{\infty} \downarrow 0$*, the sequence* $\{w^\mu\}_{\mu=0}^{\infty} \to w$ *as* $\mu \to +\infty$*, and* $\bar{y} + t^\mu w^\mu \in S$ *for each* $\mu = 1, 2, \ldots$ *. We denote this limit by* $dh(\bar{y}; w)$ *and call it derivative of* h *in direction* $w \in T_S(\bar{y})$ *at the point* $\bar{y} \in S$*.*

Remark 3.15. *Definition 3.14 is equivalent to the definition of contingent derivative for the case of a single-valued mapping (cf., for example, Aubin, 1984).*

The following statement establishes a relationship between the criterion J and the Hamiltonian function, corresponding to the optimal control problem, at a feasible process $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$. The statement essentially says that the derivative in the direction of the Bouligand tangent cone $T_{\mathbf{U}}(\bar{\mathbf{u}})$ of the criterion J, due to a perturbation in the control, can be computed as a sum of products. Each product in the sum involves the components of the partial derivative of the Hamiltonian with respect to the control variable, and the corresponding elements of the perturbation vector $\mathbf{p}$, where $\mathbf{p}$ belongs to the tangent space of the control set $\mathbf{U}$, which is defined based on the control sets U_s.

Proposition 3.16 (The Objective-Hamiltonian Relation)**.** *Let Assumptions 3.12 and 3.13 hold true, the pair* $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ *be a feasible process, the corresponding adjoint sequence* ψ *be defined by (3.13). Then for every positive integer* τ *and an index-set of non-negative integers* $\mathcal{K} := \{k_1, k_2, \ldots, k_\tau\} \subset \mathbb{N}$ *with* $k_1 < k_2 < \cdots < k_\tau$*, the following relation holds true:*

$$dJ(x_0, \bar{\mathbf{u}}; \mathbf{p}) = \sum_{i=1}^{\tau} \left(\frac{\partial}{\partial u} \mathcal{H}_{k_i}(\bar{x}_{k_i}, \bar{u}_{k_i}, \psi_{k_i+1}) \right) p_{k_i},$$

for each $\mathbf{p} := (p_0, p_1, p_2, \ldots, p_k, \ldots) \in T_{\mathbf{U}}(\bar{\mathbf{u}})$*, where*

$$\mathbf{U} := \Big\{ (u_0, u_1, u_2, \ldots, u_k, \ldots) : u_s \in U_s \text{ if } s \in \mathcal{K}; \quad u_s = \bar{u}_s \text{ if } s \notin \mathcal{K} \Big\}.$$

3.4.3 Sufficient Optimality Condition

The following theorem, which presents the main result of this section is consistent with the principles of optimal control theory, where the Hamiltonian is crucial for characterizing optimal states and control trajectories. The conditions outlined in the theorem, such as the convex and bounded criterion J, and a minimum condition related to the Hamiltonian function $\mathcal{H}$, are integral. When these conditions are met, they collectively confirm that the process $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ effectively minimizes the criterion J, thereby resolving the optimal control problem.

Theorem 3.17. *Let Assumptions 3.12 and 3.13 hold true, the pair* $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ *be a feasible process, the corresponding adjoint sequence* $\psi = \{\psi_k\}_{k=1}^{\infty}$ *be defined by (3.13), the criterion* J *be convex and there exist constants* $c, r > 0$ *such that* $J(x_0, \mathbf{u}) \leq c$ *for all* $\mathbf{u} \in r\bar{\mathbf{B}}(\bar{\mathbf{u}})$*, and let*

$$\mathcal{H}_k(\bar{x}_k, \bar{u}_k, \psi_{k+1}) \leq \mathcal{H}_k(\bar{x}_k, u, \psi_{k+1}), \quad u \in U_k \quad \text{for every } k \in \mathbb{N}. \tag{3.14}$$

Then $(\bar{\mathbf{x}}, \bar{\mathbf{u}})$ *is a global optimal process.*

Remark 3.18. *We would like to point out that the convexity assumption in the formulation of Theorem 3.17 does not imply that all functions g_k, $k \in \mathbb{N}$, are convex. Indeed, let us consider the following simple illustrative example:*

$$J(x_0, \mathbf{u}) = \left(\frac{1}{2}x_0^2 - \frac{1}{2}u_0^2\right) + \left(\frac{1}{2}x_1^2 + \frac{1}{2}u_1^2\right) \to \min,$$

subject to

$$x_{k+1} = x_k + 2u_k, \quad x_0 \in \mathbb{R}, \quad u_k \in [-1, 1], \quad k \in \mathbb{N}.$$

Here,

$$g_0(x_0, u_0) = \frac{1}{2}x_0^2 - \frac{1}{2}u_0^2$$

and

$$g_1(x_1, u_1) = \frac{1}{2}x_1^2 + \frac{1}{2}u_1^2.$$

Clearly, g_0 is not a convex function. However the criterion J is convex with respect to (u_0, u_1) because

$$\begin{aligned} J(x_0, \mathbf{u}) &= \left(\frac{1}{2}x_0^2 - \frac{1}{2}u_0^2\right) + \left(\frac{1}{2}(x_0 + 2u_0)^2 + \frac{1}{2}u_1^2\right) \\ &= x_0^2 + 2x_0u_0 + \frac{3}{2}u_0^2 + \frac{1}{2}u_1^2. \end{aligned}$$

Conclusion

The thesis has contributed to the progress regarding the optimality conditions for optimal control problems, specifically focusing on scenarios marked by uncertainty and constraints on the control variables. The work addresses robust control challenges by exploring them within the context of two-person linear-quadratic dynamic games.

Chapter 1 establishes conditions for the existence of a saddle point, also know as Nash equilibrium for linear-quadratic differential game over an infinite time horizon, especially in cases where the control actions of the minimizing player are bounded. Additionally, these findings have been adapted to the context of Stackelberg differential games. The practical applicability of these results is illustrated with an example of monetary policy.

In Chapter 2 we have established sufficient conditions for solving a linear-quadratic control problem in discrete time across an infinite horizon, where once again the controls of the minimizing player are subject to constraints. Our derivation hinges on Bellman's Principle of Optimality and involves demonstrating an equivalence between the infinite-time horizon problem and a corresponding finite-time problem. By adopting this approach, a problem defined on a finite time horizon is formulated, enabling its resolution through suitable numerical methods. Additional contributions encompass a "min-max" theorem for the unconstrained linear-quadratic problem, alongside an examination of the system's trajectory's asymptotic properties. To illustrate the practical application of our findings, we have applied these results to a model that describes the short-term dynamics of an F-16 aircraft.

In Chapter 3 we study a discrete dynamical game on an infinite-time horizon. The main result Theorem 3.11 provides a necessary condition of Pontryagin's maximum principle type. This condition is derived within the framework of a specific discrete zero-sum game formulation. Additionally, the chapter narrows its focus to a specific subset of discrete-time dynamical games that do not incorporate disturbances, examining them as a specialized case. The central result in Section 3.4.3, Theorem 3.17, introduces a novel sufficient condition for optimality. Pivotal to the proofs of both theorems is the explicit definition of the adjoint sequence ψ.

The presented example illustrates the possible practical applications.

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