Review

on the defense procedure of a dissertation thesis on the topic:
"Functionals of Levy processes and their applications"
for acquiring
the educational and scientific degree of "Doctor"
by
candidate: **Martin Jordanov Minchev**,

Field of higher education: 4. Natural sciences, mathematics and informatics.
Professional direction: 4.5. Mathematics.
Doctoral program: "Probability Theory and Mathematical Statistics",
Department: "Probabilities, Operations Research and Statistics",
Faculty of Mathematics and Informatics (FMI),
Sofia University "St. Kliment Ohridski" (SU),

The review is prepared by: **Professor Nikolay Mihaylov Yanev, PhD, D.Sc.,**
Section Operations Research, Probability and Statistics, IMI-BAS,
as a member of the scientific jury, according to Order No. RD 38-200/26.04.2024
of the Rector of Sofia University.

1. General characteristics of the dissertation work and the presented materials.

The dissertation work "Functionals of Levy processes and their applications" with author Martin J. Minchev and scientific advisors Professor DSc Mladen Savov (FMI, corresponding member of BAS) and Professor DSc Marusia Slavchova-Bozhkova (FMI), contains 175 pages of text, out of which 13 pages are bibliography. The dissertation is written in English and consists of six chapters, an Abstract, and 2 appendices. In fact, the first chapter is titled Introduction, and the sixth is Conclusion. The first three chapters are of a review nature, and the original results are presented in Chapters 4 and 5. An Abstract in English (57 pages) and in Bulgarian (56 pages) are presented with identical content. A total of 25 documents are presented as part of the procedure, certifying the legality of the procedure. Some of the most important documents include: bachelor's and master's diplomas, diploma thesis, academic

transcript, autobiography, participation in scientific projects, lists of publications and presentations, certificate of passing the doctoral minimum, etc.

2. Data and personal impressions of the candidate

Martin Vasiliev Minchev was born on 28.01.1994 in Shumen. He graduated from the Sofia Mathematical High School "Paisiy Hilendarski" in 2013. He holds a bachelor's degree in 2018 from the FMI at SU, as well as a master's degree from Sorbonne (Paris), where he studied in 2018-2019 and defended a thesis on the topic "Cutoff on Ramanujan Graphs" with supervisor Justin Salez. The autobiography also mentions that M. Minchev followed a bachelor's program in mathematics and physics at Lycee Louis-le-Grand (Paris, France) in 2014-2015. From 05.10.2020 to 05.10.2023, he was a regular doctoral student at the Department of Probabilities, Operations Research and Statistics at FMI-SU. The pre-defense of the dissertation took place on 15.04.2024 at the Department of Probabilities, Operations Research and Statistics at FMI-SU, as seen in the attached protocol. Since 2015, he has been teaching Probability and Statistics, Random Processes, Algebra, and Analysis at FMI. He is proficient in English and French. He has participated in 5 scientific projects and in 11 conferences and seminars.

I have good personal impressions of M. Minchev from the course on "Branching processes," as well as from his presentations at conferences and seminars.

3. Content analysis of the candidate's scientific and scientifically applied achievements, contained in the presented dissertation and related publications included in the procedure.

In general, the presented dissertation is dedicated to a class of random processes known as Levy processes. More specifically, it investigates exponential functionals of such processes. The research is actually focused on exponential functionals of a subclass of Levy processes called subordinators (i.e. those that are monotonous in some sense).

The first three chapters are introductory in nature and present the basic mathematical objects under investigation, as well as the most important results known about them up to this point. Chapter 1 (Introduction) introduces some of the basic concepts related to Levy processes and subordinators, such as the Wiener-Hopf factorization and so-called increasing processes. Additionally, some of the main obtained results, which will be further discussed, are briefly presented. Chapter 2 is

dedicated to different types of exponential functionals and presents some essential properties and results. Important theorems and comments related to them are provided. Furthermore, several interesting examples and applications with Markov processes, processes in random environments, financial mathematics, and others are mentioned. Chapter 3 is dedicated to the so-called Bernstein-gamma functions introduced by Pati and Savov in 2018. Some of the main obtained results, analytical properties, connections with Mellin transformations, and others are presented.

In general, this part of the dissertation (Chapters 1-3), which comprises 38 pages and has an overview character, shows that the doctoral student has delved seriously into the issues of the researched processes and (as further seen) adeptly uses this complex analytical apparatus. The two Appendices could also be included in this introductory part. Appendix A is dedicated to some interesting results about Bernstein functions, while Appendix B provides some essential results about properly changing functions.

The new results obtained are presented in Chapters 4 and 5. We will briefly focus on the most significant ones.

Chapter 4 investigates the so-called exponential functionals of a killed Levy process. This is the literal translation from English, but in Bulgarian, it would sound more euphonious as "stopped Levy process," especially since it actually refers to a "stopping time" where the random stopping time (killing) follows an exponential distribution. However, this is not essential. More importantly, the considered Levy processes are of a special type that satisfy the "Assumption (H)" (p. 41), i.e., they are subordinators without drift and with a Levy measure of positive increments (formula (14.3)). The main result for such processes is presented in Theorem 17.1, where the asymptotics of the density and its derivatives for exponential functionals of a killed Levy process are found. Three interesting consequences are also given. Consequence 17.4 examines the special case of a non-decreasing compound Poisson process. Looking only at formula (17.3), the reader might get the impression that it is the same for each n, which is not correct, as on the next page, it is seen that the constant C depends on n. Consequence 17.2 provides the asymptotics of the tail of the distribution of functionals. Section 19 presents some new results on Bernstein-gamma functions. In Lemma 19.1, the asymptotics (analogous to Stirling's asymptotics) of the ratio of the Gamma function and the Bernstein-gamma function is found, which is actually a Mellin transform for the considered "killed" functionals.

The proofs of the obtained results are presented in Section 20 and are realized over 22 pages. They mainly have an analytical character, utilize complex analysis, saddle point method, Tauberian theorems, and the entire arsenal developed in this area.

Chapter 5 consists of 85 pages and discusses issues related to exponential functionals on a deterministic horizon (i.e. for a finite moment) and in the case when their limits are infinity. Two-dimensional Bernstein-Gamma functions are used to prove these results, the study of which is of independent interest. These studies also essentially involve so-called q-potential measures. Here are some of the interesting results.

Theorem 23.1 deals with Levy processes with negative finite mathematical expectation and a properly changing tail distribution parameter greater than one. It is shown that the measure of the functional, normalized by this tail, weakly converges to a finite measure, for which a representation is found (23.2). Two interesting corollaries are also given. As an additional effect, two theorems (Theorem 23.5 and Theorem 23.6) are proven, which are well discussed in the dissertation. Section 24 examines two-dimensional Bernstein-Gamma functions. In Lemma 24.1, a connection is found between the derivatives of these functions and the derivatives of q-potential measures (24.1). Theorem 24.3 deals with Levy processes that tend to minus infinity. Equivalences between the finiteness of the moments of the Levy measure and the derivatives of the two-dimensional Bernstein-Gamma functions with respect to the first argument are found. Connections with Mellin functions and their derivatives are also found in Corollary 24.4. Theorem 24.6 proves that the densities of the convolutions of the measure exist and are locally bounded for Levy processes tending to minus infinity and having a bounded density of the q-potential measure. Statement 25.1 finds a representation for the convolutions of the potential measures. Theorem 25.3 proves that for Levy processes tending to infinity, the potential measures are Radon type and an analogue of the basic renewal theorem is valid.

The proofs of the results obtained in Chapter 5 are presented in Sections 25, 26, and 27 and occupy 64 pages. As it is well known, all information about Levy processes is contained in the Levy-Khintchine formula and has mainly an analytical character. It is amazing how many interesting results are obtained from it. Above all, the author shows that he has seriously entered into this topic and skillfully uses the developed analytical apparatus related to Wiener-Hopf factorizations, Bernstein-Gamma functions, integral transformations of Mellin and Laplace, Tauberian theorems, use of a number of facts from complex analysis, and more. In Chapter 6 (Conclusions), the author lists (quite modestly) the main results of the dissertation in 4 points.

Undoubtedly, the results obtained in the dissertation are original and deserve a high evaluation.

4. Approval of the Results

The main results of the dissertation have been published in the well-known international journal Bernoulli (2023) with an impact factor of 1.393. The work is joint with the scientific advisor M. Savov. In addition, these results have been reported at five conferences and seminars. The most positive aspect is that they have already been cited in 7 publications by authoritative scientists in prestigious international journals with an impact factor.

The dissertation meets the minimum national requirements (according to Art. 2b, para. 2 and 3 of the Higher Education Act) and the additional requirements of Sofia University "St. Kliment Ohridski" for obtaining an educational and scientific degree of "Doctor" in the scientific field and professional direction of the procedure.

There is no proven plagiarism in the presented dissertation work and scientific papers through this procedure.

5. Qualities of the abstract

The abstract (in Bulgarian and English) meets all the requirements for its preparation and accurately presents the results and content of the dissertation work.

6. Critical notes and recommendations

The dissertation and abstract are written carefully with the necessary accuracy and completeness of the results, but there are some inaccuracies that can be considered technical in nature. Here are some of them:

1) on page 7, line 5, the measure Π is incorrectly defined, what is given there is actually the tail of the measure;

2) on page 7, last line, the definition of the reverse process of local time (the process of increasing times) needs correction;

3) on page 8, formula (3.1), the probability P in the second row is probably unnecessary;

4) on page 74, line 6, the Γ-function in the asymptotics is unnecessary;

5) on page 154, 10th line from the bottom, in the derivative formula, instead of q, d should be used.

As already noted, the dissertation writer's literary awareness is at a high level, as well as the methodological level. All this provides a basis for recommending that the dissertation writer continue their research in this area.

7. Conclusion

As already noted, the analysis of the presented dissertation shows that the results obtained are of a high scientific level and have already been seriously evaluated by specialists in the field of Levy processes. The dissertation writer skillfully handles a complex mathematical apparatus and has successfully entered this field.

Having reviewed the dissertation work presented in the procedure and the accompanying scientific papers and based on the analysis of their significance and the scientific and scientifically applicable contributions contained in them, I confirm that the presented dissertation work and scientific publications, as well as the quality and originality of the results and achievements presented in them, meet the requirements of the Higher Attestation Commission for Scientific Research, its implementing regulations, and the corresponding Regulations of Sofia University "St. Kliment Ohridski" for acquiring the educational and scientific degree of "Doctor" in scientific field 4. Natural Sciences, Mathematics and Informatics, and professional direction 4.5. Mathematics. In particular, the candidate meets the minimum national requirements in the professional field and no plagiarism has been found in the scientific papers submitted for the competition.

Based on the above, I recommend to the scientific jury to award **Martin Yordanov Minchev** the educational and scientific degree "**Doctor**" in scientific field 4. Natural Sciences, Mathematics and Informatics, professional direction 4.5. Mathematics.

July 1, 2024 Prepared the review: .....

(Professor Nikolay M. Yanev, PhD, D.Sc.)