REVIEW

under the procedure for acquisition of the educational and scientific degree "Doctor"
by candidate Martin Yordanov Minchev**,**
of the PhD Thesis entitled: "Functionals of Levy processes and their applications",
In the Scientific field: **4. Natural Sciences, Mathematics and Informatics**
Professional field: **4.5. Mathematics**
Doctoral program "Theory of Probability and Mathematical Statistics", Department „Probability, Operation Research and Statistics",
Faculty of Mathematics and Informatics (FMI), **Sofia University "St. Kliment Ohridski" (SU),**

The review has been prepared by: **Associate Professor, Ph.D. Doncho Donchev**,
as a member of the scientific jury for the defense of this PhD thesis according to Order № RD 38-200/ 26.04.2024 y of the Rector of the Sofia University.

1. **General characteristics of the dissertation thesis and the presented materials**

The thesis is of 175 pages, including 6 chapters, two appendices and a list of cited literary sources. The latter contains 178 entries.

2. Short CV and personal impressions of the candidate

I have known Martin Minchev being a member of the examination jury of his candidate-doctoral examination, as well as his two examinations in the specialty during his doctoral studies. In addition, he was my assistant in 2023 when I was teaching a course of Mathematical Statistics for Data Analysis students. My opinion is that he has brilliant mathematical accomplishment combined with hard work and determination.
The subject of his dissertation is close to me, since in 2018 I was a reviewer of his supervisor's doctoral thesis, dedicated to the same problem. Comparing the two dissertations, I am impressed how substantial is the progress in his thesis in terms of new results, ideas, and methodology.

3. Content analysis of the scientific and applied achievements of the candidate, contained in the presented PhD thesis and the publications to it, included in the procedure

The theory of exponential functionals of Lévy processes has more than 50 years long history, and it is developing intensively in the last years. The reason is that it itself is extremely beautiful and deep being a crossroad of different fields of mathematics, as well as the numerous and important applications it finds in different areas of science (branching processes in a random environment, killing times of self-similar processes, pricing of Asian option, etc.).

In its early stages the theory of exponential functionals exploits mainly probabilistic methods. In his thesis the author mentions two well-known approaches. The first approach, which originates to John Lamperti, is based on the relation between the exponential functional and the entrance law of a self-similar Markov process (the cited Theorem 4.3 in the dissertation), which allows, in some cases, to find the distribution of the functional. The second approach is based on the observation that any Lévy process can be represented as a logarithm of a stochastic exponential of a generalized Ornstein-Ollenbeg process, and therefore, the distribution of its exponential functional can be described in terms of the stationary meaasure of this process.

Recently, thanks to the ideas of Pierre Patie and Mladen Savov, which are in the focus in this thesis, has been developed a new approach to the theory of exponential functionals. Its idea is the following. It is well-known that the characteristic exponent contains all information about the Lévy process. The idea of P. Patie and M. Savov is to find a direct connection between the analytical properties of the characteristic exponent and the subtle probabilistic properties of the distribution of the exponential functional, i.e. the asymptotic of the tail of the distribution, properties of the moments, the smoothness of the density etc. However, the implementation of this approach requires using of heavy analytical tools from different fields of mathematics (complex analysis, integral transformations, Tauber approximations, etc.). The authors revealed the important role of the Bernstein-Gamma functions and their connection with the distribution of the exponential functional. In my opinion, the meaning of the words "the characteristic exponent contains all the information about the Lévy process" is revealed thanks to this approach.

In his thesis Martin Minchev is developing this approach in the direction described above. The author has shown deeply understanding of theory of exponential functionals, as well as fluency in different fields of mathematics, necessary for studying the analytical properties of the characteristic exponent. The main idea of the approach is the following. The starting point is an equation which satisfy the complex moments of the exponential functional, i.e. Mellin's transform of its probability density. This equation is similar to the functional equation of the Bernstein-Gamma functions. The only difference that the role of the Bernstein function is played by the function $-z/\Psi(-z)$, where $\Psi(z)$ is the characteristic exponent of the Lévy process. On the other hand, the inversion formula, namely the Mellin–Barnes integral, allows to represent the probability density of the exponential functional in terms its Mellin's transform, which in turn depends on the characteristic exponent $\Psi(z)$, being a solution to the functional equation. This relationship between $\Psi(z)$ and the probability density of the functional is extremely subtle, and in order to extract an useful information from it one should overcome essential difficulties. I will mention some of them:

1) The characteristic exponent $\Psi(z)$ is not a Bernstein function. Such functions are the factors of its Wiener-Hopf factorization, and in the representation of Mellin's transform of the functional one should work with the Bernstein-Gamma functions corresponding to them;

2) Before studying general Levy processes it is worth to consider some special cases, i.e. subordinators, as well as functionals with a simpler structure (a random exponentially distributed horizon (incl. infinite), etc.);

3) The next step is to make transition to general Lévy processes and to functionals with a deterministic horizon.

The main contributions of the author are as follows:

1) In case of subordinators, have been obtained results concerning the smoothness and the tail behavior of the p.d.f. of the functional;
2) For Lévy processes with negative expectations $E\xi_1<0$ and a properly varying tail of the function $P(\xi_1>t)$, has been proved an weak convergence of their appropriately scaled exponential functionals to some finite measure which is characterized in terms of both the Bernstein-Gamma functions corresponding to the Wiener-Hopf factors and the q-potential measure of the process;
3) Have been obtained improved Stirling asymptotics of a certain class of Bernstein-Gamma functions;
4) Has been found a relation between the bivariate Bernstein-Gamma functions and the convolutions of the q-potential measure of the process.

The main results of the thesis are stated and proved in Chapters 4 and 5.

In Chapter 4 are studied exponential functionals of subordinators ξ_t such that $\lim_{t\to\infty}\xi_t=\infty$. Provided that the Lévy measure of these processes satisfies the positive growth condition, it is shown that the distribution of the corresponding functional has derivatives of any order, and an exact asymptotic formula for their tail behavior is obtained. The later is in terms of the Bernstein function of the subordinator, more precisely of a function that functionally depends on it (Theorem 17.1). This result is applied for evaluating the tail distribution of the functional (Corollary 17.2). Special attention is paid to the case of a compound Poisson process with rare large jumps (Corollary 17.4), as well as to α-stable subordinators and Gamma subordinators.

Chapter 5 is devoted to exponential functionals with a deterministic horizon corresponding to Levy processes ξ_t such that $\lim_{t\to\infty}\xi_t=-\infty$ and $E\xi_1<0$. This problem is closely related to the problem of the distribution of functionals of exponentially killed Lévy processes, since, if the latter is solved, then the distribution of the functional with a deterministic horizon can be found by inverting a Laplace transform w.r.t. the killing rate. The Wiener-Hopf factorization of the characteristic exponent of such processes has two nontrivial factors, which results in the appearance of bivariate Bernstein and Bernstein-Gamma functions. Similarly to the case of a subordinator tending to ∞ as $t\to\infty$, Mellin's transform of the functional of the exponentially killed Levy process satisfies a functional equation w.r.t. the spatial variable, and admits representation in terms of bivariate Bernstein-Gamma functions.

The main result in this chapter is Theorem 25.1, where is shown that the distribution of the suitably normalized functional $I_\xi(t)$ converges weakly to a finite measure characterized in terms of the Bernstein-Gamma functions corresponding to the factors of the Wiener-Hopf factorization. In Theorem 25.6 is shown that important corollaries of Theorem 25.1 still hold true under conditions imposed directly on the Lévy measure of the process rather than the condition $E|\xi_1|<\infty$. In Lemma 26.1, the Bernstein-Gamma functions in the formulation of Theorem 25.1 are characterized in terms of the q-potential measure of the Lévy process. In fact, this result in principle solves the problem of the distribution of both functionals of exponentially killed Lévy processes and functionals with a deterministic horizon. This lemma is used for obtaining an important information about the properties of the Bernstein-Gamma functions corresponding to the Wiener-Hopf factors.

In the proof of the above results was used a rich mathematical toolkit in order to overcome considerable technical difficulties, requiring the use of a large number of additional lemmas and auxiliary results, many of them new and of independent interest.

4. Approbation of the results

The results from Chapter 4 of the thesis have been published in Bernulli (Q1, joint work with the supervisor). The results of Chapter 5 have been reported at 5 conferences, two of which are abroad. The published paper has 7 citations in journals with a high impact factor. I assume that the contribution of both authors is equal.

My opinion is that:

a) the thesis meets the minimum national requirements (according to Article 2b, Paragraphs 2 and 3 of the RSARB) and, accordingly, the additional requirements of SU "St. Kliment Ohridski" for the acquisition of an educational and scientific degree "doctor" in the scientific field and professional direction of the procedure;

b) the results in the thesis and related published articles do not repeat those from previous procedures for acquiring a scientific title and academic position;

c) there is no plagiarism in the submitted dissertation and all scientific works under this procedure.

5. Qualities of the abstract

The abstract meets all the requirements for its preparation and correctly presents the results and content of the thesis.

6. Critical notes and recommendations- no remarks

7. Conclusion

Having become acquainted with the PhD thesis presented in the procedure and the accompanying scientific papers and on the basis of the analysis of their importance and the scientific and applied contributions contained therein, **I confirm** that the presented PhD thesis and the scientific publications to it, as well as the quality and originality of the results and achievements presented in them, meet the requirements of the ADAS in the Republic of Bulgaria, the Rules for its Implementation and the corresponding Rules at the Sofia University "St. Kliment Ohridski" (FMI-SU) for acquisition by the candidate of educational and scientific degree "Doctor" in the Scientific field **4. Natural Sciences, Mathematics and Informatics**, Professional field **4.5. Mathematics.** In particular, the candidate meets the minimal national requirements in the professional field and no plagiarism has been detected in the scientific papers submitted for the competition.

Based on the above, **I strongly recommend** the scientific jury to award to Martin Yordanov Minchev, the educational and scientific degree "Doctor" in the Scientific field **4. Natural Sciences, Mathematics and Informatics**, Professional field **4.5. Mathematics**. (Theory of Probability and Mathematical Statistics).

Date: 15.06.2024

Reviewer: ....
/Associate Professor Doncho Donchev, PhD/