SU “St. Kliment Ohridski”
Faculty of Mathematics and Informatics
SUMMIT, Contract 70-123-195/12.02.2024

On invariant polynomials in free associative algebras over a field of arbitrary characteristic

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23 May 2024

Joint Project with Vesselin Drensky,
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Joint Project with Vesselin Drensky, Sehmus Findik

Invariant theory

Rota, GC. (2001). What is invariant theory, really? In: Crapo, H., Senato, D. (eds) Algebraic Combinatorics and Computer Science. Springer, Milano. https://doi.org/10.1007/978-88-470-2107-5_4

Rota (2001)

"Invariant theory is the great romantic story of mathematics."

"In our century, Lie theory and algebraic geometry, differential algebra and algebraic combinatorics are offsprings of invariant theory. "

"Like the Arabian phoenix arising from its ashes, classical invariant theory, once pronounced dead, is once again at the forefront of mathematics."

Invariant theory

Rota (2001)

“A pedestrian definition of invariant theory might go as follows: invariant theory is the study of orbits of group actions. Such a definition is correct, but it must be supplemented by a programmatic statement. Hermann Weyl, in the introduction to his book The Classical Groups, was the first in this century to give a sweeping overview of the program of invariant theory. He summarized this program in two basic assertions. The first states that "all geometric facts are expressed by the vanishing of invariants", and the second states that "all invariants are invariants of tensors".”

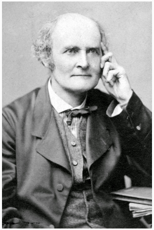
“The program of invariant theory, from Boole to our day, is precisely the translation of geometric facts into invariant algebraic equations expressed in terms of tensors.”

Origins of Invariant Theory

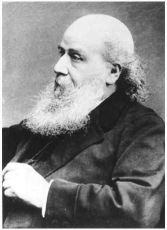
- Classically, invariant theory deals with polynomial functions, which do not change under linear transformations.
- The origins of the theory can be found to the works of Lagrange (1770's) and Gauss (early 1800's) who studied the representation of integers by quadratic binary forms and used the discriminant to distinguish nonequivalent forms.
- The real invariant theory began with the works of George Boole and Otto Hesse in the 1840's.
- Originally efforts were focused on describing properties of polynomials by vanishing of invariants, but shifted towards finding fundamental sets of invariants.
- Later, the further development of the theory continued in the work of a pleiad of distinguished mathematicians, among them Cayley, Sylvester, Clebsch, Gordan (known as "König der Invariantentheorie"), and Hilbert.

Mathematicians who worked in the field

O. Hesse



A. Cayley



J. Sylvester

D. Hilbert

A. Clebsch

P. Gordan

Algebra of invariants

- Let K be a field of arbitrary characteristic.
- Let $K[X_d]$ be the polynomial algebra in d variables over a field K.
- Let $K\langle X_d\rangle$ be the free associative algebra freely generated by the set $X_d = \{x_1, \ldots, x_d\}$, $d \geq 2$.

Every mathematics student knows the Fundamental theorem of symmetric polynomials

Every symmetric polynomial can be expressed in a unique way as a polynomial of the elementary symmetric polynomials.

More precisely: We fix a field K, a set of d variables $X_d = \{x_1, \dots, x_d\}$ and consider the polynomial algebra $K[X_d] = K[x_1, \dots, x_d]$. We define an action of the symmetric group S_d on $K[X_d]$ by

$$\sigma : f(x_1, \dots, x_d) \to f(x_{\sigma(1)}, \dots, x_{\sigma(d)}), \quad \sigma \in S_d, f \in K[X_d].$$

Then:

(1) *The algebra of symmetric polynomials*

$$K[X_d]^{S_d} = \{f(X_d) \in K[X_d] \mid \sigma(f) = f \text{ for all } \sigma \in S_d\}$$

is generated by

$$e_1 = x_1 + \cdots + x_d = \sum_{i=1}^{d} x_i,$$

$$e_2 = x_1x_2 + x_1x_3 + \cdots + x_{d-1}x_d = \sum_{i<j} x_i x_j,$$

$$\cdots$$

$$e_d = x_1 \cdots x_d;$$

(2) *If* $f \in K[X_d]^{S_d}$, *then there exists a unique polynomial* $p \in K[y_1, \ldots, y_d]$ *such that* $f = p(e_1, \ldots, e_d)$. *In other words, the elementary symmetric polynomials are algebraically independent.*

Invariant theory studies the following generalization:

Let $\mathrm{char}(K) = 0$. The group $GL_d(K)$ of $d \times d$ invertible matrices acts canonically from the left on the vector space with basis $X_d = \{x_1, \dots, x_d\}$. This action is extended diagonally on $K[X_d]$ by the rule

$$g(f(x_1, \dots, x_d)) = f(g(x_1), \dots, g(x_d)), \quad g \in GL_d(K), f \in K[X_d].$$

Definition

If G is a subgroup of $GL_d(K)$, then **the algebra of G-invariants** *is*

$$K[X_d]^G = \{f \in K[X_d] \mid g(f) = f \text{ for all } g \in G\}.$$

Remark

Usually one considers another action of $GL_d(K)$ and assumes that $GL_d(K)$ acts on a d-dimensional vector space V_d with basis $\{v_1, \dots, v_d\}$. Then one defines the polynomial functions

$$x_i : V_d \to K, \quad i = 1, \dots, d,$$

by

$$x_i(\xi_1 v_1 + \cdots + \xi_d v_d) = \xi_i, \quad \xi_1, \dots, \xi_d \in K.$$

If $f(X_d) \in K[X_d]$ and $g \in GL_d(K)$, then

$$g(f) : v \to f(g^{-1}(v)), \quad v \in V_d.$$

Remark

Both ways do not differ essentially. The group $GL_d(K)$ and its opposite $GL_d(K)^{\mathrm{op}}$ (acting on V_d from the right) are isomorphic by

$$GL_d(K) \ni g \to (g^t)^{-1} \in GL_d(K)^{\mathrm{op}},$$

where g^t is the transpose of g, and then the "classical" action of $GL_d(K)$ on the polynomials considered as functions is the same as our "diagonal" action induced by the canonical action from the left of $GL_d(K)$ on the vector space with basis X_d.
For our generalizations it is more convenient to consider our action of $GL_d(K)$.

Problem: Describe $K[X_d]^G$

(1) *Is the algebra $K[X_d]^G$ finitely generated for all subgroups G of $GL_d(K)$?*

This is the main motivation for the 14-th problem of Hilbert from the International Congress of Mathematicians in Paris in 1900.

Answers.

- G – finite – **YES** (Emmy Noether (1915, 1926));

Answers.

Der Endlichkeitssatz der Invarianten endlicher Gruppen.

Von

Emmy Noether in Erlangen.

Im folgenden soll ein ganz elementarer — nur auf der Theorie der symmetrischen Funktionen beruhender — Endlichkeitsbeweis der Invarianten *endlicher* Gruppen gebracht werden, der zugleich eine *wirkliche Angabe des vollen Systems* liefert; während der gewöhnliche, auf das Hilbertsche Theorem von der Modulbasis (Ann. 36) sich stützende Beweis nur Existenzbeweis ist.*)

Die endliche Gruppe $\mathfrak{H}$ bestehe aus den h linearen Transformationen (von nichtverschwindender Determinante) $A_1 \cdots A_h$, wobei durch A_k die lineare Transformation

$$x_1^{(k)} = \sum_{\nu=1}^{n} a_{1\nu}^{(k)} x_\nu, \cdots, x_n^{(k)} = \sum_{\nu=1}^{n} a_{n\nu}^{(k)} x_\nu$$

oder abkürzend: $(x^{(k)}) = A_k(x)$ dargestellt sei. Die Gruppe $\mathfrak{H}$ führt also die Reihe (x) mit den Elementen $x_1 \cdots x_n$ über in die Reihen $(x^{(k)})$ mit den Elementen $x_1^{(k)} \cdots x_n^{(k)}$. Da unter $A_1 \cdots A_h$ die Identität enthalten sein muß, ist auch unter den Reihen $(x^{(k)})$ die Reihe (x) enthalten. — Unter einer ganzen rationalen (absoluten) Invariante der Gruppe sei eine solche ganze rationale Funktion von $x_1 \cdots x_n$ verstanden, die bei Anwendung von $A_1 \cdots A_h$ identisch ungeändert bleibt; für eine solche Invariante $f(x)$ gilt also:

$$(1) \qquad f(x) = f(x^{(1)}) = \cdots = f(x^{(h)}) = \frac{1}{h} \cdot \sum_{k=1}^{h} f(x^{(k)}).$$

Emmy Noether

Answers.

Endlichkeitssatz of Emmy Noether, 1916.

Let K be a field of characteristic 0 and G be a finite subgroup of $\mathrm{GL}_d(K)$. Then the algebra of invariants $K[X_d]^G$ is finitely generated and has a system of generators $f_1, \ldots, f_m$, where each f_i is homogeneous polynomial of degree bounded by the order of the group G.

For reductive groups and characteristic 0, the proof is also contained in Hilbert's work.
Emmy Noether also gave proof for fields of any characteristic in 1926.

Answers.

- G-reductive (in some sense "nice") – **YES** (Although not stated in this generality, the (nonconstructive) proof is contained in the work of Hilbert from 1890–1893);
- In the general case – **NO** (the counterexample of Nagata in the 1950s).

Counterexample for infinite groups in 1959

ON THE 14-TH PROBLEM OF HILBERT.*[1]

To Professor Oscar Zariski on his sixtieth birthday.

By Masayoshi Nagata.

The following problem is known as the 14-th problem of Hilbert:

Let k be a field and let $x_1, \cdots, x_n$ be algebraically independent elements over k. Let K be a subfield of $k(x_1, \cdots, x_n)$ containing k. Is $k[x_1, \cdots, x_n] \cap K$ finitely generated over k?

The purpose of the present paper is to answer the question in the negative by giving a counter-example. In fact, we shall give a counter-example to the following restricted case, which was the original question of Hilbert, and which we shall call the *original 14-th problem:*

Let G be a subgroup of the full linear group of $k[x_1, \cdots, x_n]$ and let $\mathfrak{o}$ be the set of elements of $k[x_1, \cdots, x_n]$ which are invariant under G. Is $\mathfrak{o}$ finitely generated over k?

We shall note that the construction of our example is *independent of the characteristic* (and k may be the field of complex numbers).

Masayoshi Nagata

Problem: Describe $K[X_d]^G$

(2) If $K[X_d]^G$ is generated by $f_1, \dots, f_m$, then it is a homomorphic image of $K[Y_m]$ ($\pi : K[Y_m] \to K[X_d]^G$ is defined by $\pi(y_j) = f_j$). **Find generators of the ideal** $\ker(\pi)$.

Answers. Explicit sets of generators for different groups G.

Hilbert's Basissatz. *Every ideal of $K[Y_m]$ is finitely generated.* (Nonconstructive proof.)

Chevalley-Shephard-Todd

Theorem (Chevalley-Shephard-Todd)

For G finite $K[X_d]^G \cong K[Y_d]$ if and only if $G < GL_d(K)$ is generated by pseudo-reflections (matrices of finite multiplicative order with all eigenvalues except one equal to 1 or matrices of finite multiplicative order that fix a hyperplane).

$K[X_d]$ and $K[X_d]^G$ are graded

Definition.

A ring R is said to be *graded*, if it can be decomposed as direct sum

$$R = \bigoplus_{i=0}^{\infty} R_i$$

of additive groups, such that $R_i R_j \subseteq R_{i+j}$.
An algebra A is said to be graded if it is graded as a ring.

For the polynomial algebra and algebra of invariants, there is the natural grading

$$K[X_d] = \bigoplus_{k\geq 0} (K[X_d])^{(k)} \text{ and } K[X_d]^G = \bigoplus_{k\geq 0} (K[X_d]^G)^{(k)}.$$

Theorem (Hilbert-Serre).

The Hilbert series $H(K[X_d]^G, t) = \sum_{n=0}^{\infty} \dim(K[X_d]^G)^{(n)} t^n$ is a rational function of t in the form

$$\frac{f(t)}{\prod_{i=1}^{s}(1-t^{k_i})}, \quad f(t) \in \mathbb{Z}[t].$$

Theorem (Molien Formula, 1897).

For a finite group G,

$$H(K[X_d]^G, t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1-gt)}.$$

Noncommutative generalizations

Problem

Replace the polynomial algebra $K[X_d]$ with another noncommutative algebra which shares many of the properties of $K[X_d]$.

The most natural candidate is the free associative algebra $K\langle X_d\rangle$ (or the algebra of polynomials in d noncommuting variables). This algebra has the same **universal property** as $K[X_d]$:

- *If R is a commutative algebra, then every mapping $X_d \to R$ can be extended in a unique way to a homomorphism $K[X_d] \to R$.*
- *If R is an associative algebra, then every mapping $X_d \to R$ can be extended in a unique way to a homomorphism $K\langle X_d\rangle \to R$.*

Symmetric polynomial in $K\langle X_d\rangle$

Problem

Describe the symmetric polynomials in $K\langle X_d\rangle$.

Answer - M.C. Wolf, Symmetric functions of non-commutative elements, *Duke Math. J.* 2 (1936), No. 4, 626-637.

Next step

Develop noncommutative invariant theory and study $K\langle X_d\rangle^G$.

Go further

Study $F(X_d)^G$, where $F(X_d)$ is an algebra with universal property similar to those of $K[X_d]$ and $K\langle X_d\rangle$ (the free Lie algebra $L(X_d)$, the free nonassociative algebra $K\{X_d\}$, the relatively free algebra $F_d(\mathfrak{V})$ of a variety of algebras $\mathfrak{V}$).

The main results of Margarete Wolf

Theorem

(i) *The algebra of symmetric polynomials* $K\langle X_d\rangle^{\mathrm{Sym}(d)}$, $d \geq 2$, *is a free associative algebra over any field* K.

(ii) *It has a homogeneous system of free generators* $\{f_j \mid j \in J\}$ *such that for any* $n \geq 1$ *there is at least one generator of degree* n.

(iii) *The number of homogeneous polynomials of degree* n *is the same in every homogeneous free generating system.*

(iv) *If* $f \in K\langle X_d\rangle^{\mathrm{Sym}(d)}$ *has the presentation*

$$f = \sum_{j=(j_1,\ldots,j_m)} \alpha_j f_{j_1} \cdots f_{j_m}, \quad \alpha_j \in K,$$

then the coefficients α_j *are linear combinations with integer coefficients of the coefficients of* $f(X_d)$.

Symmetric polynomials in two noncommuting variables

Theorem (Wolf)

In the free generating set of $K\langle X_2\rangle^{S_2}$ there is precisely one element of degree n for each $n \geq 1$.

What happened with noncommutative symmetric polynomials after Margarete Wolf?

- Symmetric functions in commuting variables are studied from different points of view. The same have happened in the noncommutative case. In her paper Margarete Wolf studied the algebraic properties of $K\langle X_d\rangle^{S_d}$.

- The next result in this direction appeared more than 30 years later in

  G.M. Bergman, P.M. Cohn, Symmetric elements in free powers of rings, *J. Lond. Math. Soc., II. Ser. 1 (1969), 525-534* where the authors generalized the main result of Wolf.

There is an enourmous literature devoted to different aspects in the theory. We shall mention few papers and one book only.

- **I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh, J.-Y. Thibon,** Noncommutative symmetric functions, *Adv. Math.* 112 (1995), No. 2, 218-348.
- **S. Fomin and C. Greene**, Noncommutative Schur functions and their applications, *Discrete Math.* 193 (1998), 179-200.
- **M.H. Rosas, B.E. Sagan**, Symmetric functions in noncommuting variables, *Trans. Am. Math. Soc.* 358 (2006), No. 1, 215-232.
- **M.H. Rosas, B.E. Sagan**, Symmetric functions in noncommuting variables, *Trans. Am. Math. Soc.* 358 (2006), No. 1, 215-232.
- **N. Bergeron, C. Reutenauer, M. Rosas, M. Zabrocki**, Invariants and coinvariants of the symmetric groups in noncommuting variables, *Canad. J. Math.* 60 (2008), No. 2, 266-296.
- **D.S. Kaliuzhnyi-Verbovetskyi, V. Vinnikov**, Foundations of Free Noncommutative Function Theory, Mathematical Surveys and Monographs, vol. 199, Providence, RI, American Mathematical Society, 2014.

Noncommutative invariant theory

- Let K be a field with arbitrary characteristic.
- As in the commutative case we assume that the general linear group $GL_d(K)$ acts on the vector space with basis X_d and extend this action diagonally on $K\langle X_d\rangle$ by the rule

$$g(f(x_1,\ldots,x_d)) = f(g(x_1),\ldots,g(x_d)), \quad g\in GL_d(K), f\in K\langle X_d\rangle.$$

- If G is a subgroup of $GL_d(K)$, then **the algebra of G-invariants** is

$$K\langle X_d\rangle^G = \{f\in K\langle X_d\rangle \mid g(f) = f \text{ for all } g\in G\}.$$

Similarity and differences between commutative and noncommutative invariant theory

The first natural questions are:

- *Which results in commutative invariant theory hold also in the noncommutative case?*
- *Which results are not true?*

The problem for finite generation

- The group $G \subset GL_d(K)$ acts on the vector space with basis X_d by **scalar multiplication** if G consists of scalar matrices.
- If G is finite and acts by scalar multiplication, then G is cyclic. If $|G| = q$ then $K\langle X_d\rangle^G$ is generated by all monomials of degree q. The number of such monomials is equal to d^q and hence the algebra $K\langle X_d\rangle^G$ is isomorphic to the free algebra $K\langle Y_{d^q}\rangle$.

Koryukin, Dicks and Formanek, Kharchenko

It has turned out that the analogue of the theorem of **Emmy Noether** for the finite generation of $K[X_d]^G$ for finite groups G holds for $K\langle X_d\rangle^G$ in this very special case only.

Theorem (Koryukin, Dicks and Formanek, Kharchenko)

Let G be a finite subgroup of $GL_d(K)$. Then $K\langle X_d\rangle^G$ is finitely generated if and only if G acts on the vector space with basis X_d by scalar multiplication.

W. Dicks, E. Formanek, Poincaré series and a problem of S. Montgomery, *Lin. Multilin. Algebra* 12 (1982), 21-30.

V.K. Kharchenko, Noncommutative invariants of finite groups and Noetherian varieties, *J. Pure Appl. Algebra* 31 (1984), 83-90.

Their results were generalized in 1984 for infinite groups.

Theorem (Koryukin)

Let G be an arbitrary (possibly infinite) subgroup of the matrix group $\mathrm{GL}_d(K)$. Let KY_m be a minimal (with respect to inclusion) vector subspace of KX_d such that $K\langle X_d\rangle^G \subseteq K\langle Y_m\rangle$. Then $K\langle X_d\rangle^G$ is finitely generated if and only if G acts on KY_m as a finite cyclic group of scalar matrices.

Koryukin, A. N. Noncommutative invariants of reductive groups. Algebra Logika 23, 4 (1984), 419-429. Translation: Algebra and Logic 1984; 23

Koryukin's 1984 Paper

УДК 519.48

О НЕКОММУТАТИВНЫХ ИНВАРИАНТАХ РЕДУКТИВНЫХ ГРУПП

А. Н. КОРЮКИН

В настоящей работе рассматривается вопрос о конечной порождаемости алгебр инвариантов некоторых линейных групп, действующих на конечно-порожденных ассоциативных алгебрах. При стандартной постановке вопроса в некоммутативном случае уже для конечных групп получаются в основном отрицательные результаты. В этом случае справедлива следующая теорема, доказанная независимо Диксом и Формaнеком [1] и Харченко [2]:

ТЕОРЕМА. Пусть G – конечная группа линейных преобразований конечномерного пространства V. Рассмотрим индуцированное действие G на тензорной алгебре $F\langle V\rangle$ пространства V. Тогда алгебра инвариантов $F\langle V\rangle^G$ конечно-порождена в том и только в том случае, когда G – группа скалярных преобразований.

Finite generation with additional action

Theorem (Koryukin)

Let the symmetric group S_n of degree n, $n = 1, 2, \ldots$, act from the right on the homogeneous elements of degree n in $K\langle X_d\rangle$ by the rule

$$(x_{i_1} \cdots x_{i_n}) \circ \sigma^{-1} = x_{i_{\sigma^{-1}(1)}} \cdots x_{i_{\sigma^{-1}(n)}}, \quad \sigma \in S_n.$$

We equip the algebra $K\langle X_d\rangle$ with this additional action and denote it $(K\langle X_d\rangle^G, \circ)$ - an S-algebra.

Let the field K be arbitrary and let G be a reductive subgroup of $\mathrm{GL}_d(K)$ (i.e. all rational representations of G are completely reducible). Then the S-algebra $(K\langle X_d\rangle^G, \circ)$ (with this additional action) is finitely generated.

A.N. Koryukin, Noncommutative invariants of reductive groups (Russian), *Algebra i Logika* 23 (1984), No. 4, 419-429. Translation: Algebra Logic 23 (1984), 290-296.

What happens with the Chevalley-Shephard-Todd theorem

Theorem. (Lane, Kharchenko)

Let G be a finite subgroup of $GL_d(K)$. Then the algebra of noncommutative G-invariants $K\langle X_d\rangle^G$ is free.

D.R. Lane, Free Algebras of Rank Two and Their Automorphisms, *Ph.D. Thesis, Bedford College, London*, 1976.

V.K. Kharchenko, Algebra of invariants of free algebras (Russian), *Algebra i Logika* 17 (1978), 478-487. Translation: Algebra and Logic 17 (1978), 316-321.

Analogue for Molien's Formula

Molien's formula has a complete analogue in the noncommutative case, it is obtained by changing

Theorem (Dicks and Formanek, 1982).

If $G \subseteq \mathrm{GL}_d(K)$ is a finite group and the field K has characteristic 0, then the Hilbert series can be calculated by

$$H\left(K\langle X_d\rangle^G, t\right) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{1 - tr(g)t}.$$

By the **Maschke theorem** if the field K is of characteristic 0 or of characteristic $p > 0$ and p does not divide the order of G, then the finite dimensional representations of G are completely reducible. Hence this inspires the following problem.

Problem.

Let G be a finite subgroup of $\mathrm{GL}_d(K)$ and let $\mathrm{char}\,(K) = 0$ or $\mathrm{char}\,(K) = p > 0$ and p does not divide the order of G.

(i) *For a minimal homogeneous generating system of the S-algebra $(K\langle X_d\rangle^G, \circ)$ is there a bound of the degree of the generators in terms of the order $|G|$ of G, the rank d of $K\langle X_d\rangle$ and the characteristic of K?*

(ii) *Find a finite system of generators of $(K\langle X_d\rangle^G, \circ)$ for concrete groups G.*

(iii) *If the commutative algebra $K[X_d]^G$ is generated by a homogeneous system $\{f_1, \ldots, f_m\}$, can this system be lifted to a system of generators of $(K\langle X_d\rangle^G, \circ)$?*

Remark

By the **Endlichkeitsatz of Emmy Noether** if $\mathrm{char}\,(K) = 0$, then $K[X_d]^G$ has a set of generators of degree $\leq |G|$ for any finite group G.

By a result of **Fleischmann (2000)** and **Fogarty (2001)** the same upper bound holds if $\mathrm{char}\,(K) = p > 0$ does not divide the order $|G|$ of G.

Hence in **Problem (i)** it is reasonably to restrict our attention to the order of G and the rank d of $K\langle X_d\rangle$.

Let λ be the partition of n, i.e.

$$\lambda = (\lambda_1, \dots, \lambda_d).$$

We denote

$$p_\lambda = \sum x_1^{\lambda_1} \cdots x_d^{\lambda_d}.$$

In particular,

$$p_{(n)} = x_1^n + \cdots + x_d^n, \quad n = 1, 2, \dots,$$

are the power sums and

$$p_{(1^n)} = \sum_{\sigma \in \mathrm{Sym}(d)} x_{\sigma(1)} \cdots x_{\sigma(n)}, \quad n \leq d,$$

are the noncommutative analogues of the elementary symmetric polynomials.

TJM - S.B., Drensky, Dzhundrekov, Kassabov

Lemma

Over any field K of arbitrary characteristic the S-algebra $(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)$ is generated by the power sums $p_{(m)}$, $m = 1, 2 \ldots$.

Theorem

Let $\mathrm{char}\,(K) = 0$ or $\mathrm{char}\,(K) = p > d$. Then the algebra $(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)$ of the symmetric polynomials in d variables is generated as an S-algebra by the elementary symmetric polynomials $p_{(1^i)}$, $i = 1, \ldots, d$.

Newton formulas:

$$ke_k = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i} p_i$$

Noncommutative analogue of the Newton formulas:

Lemma

In $K\langle X_d \rangle$

$$k! p_{(k)} + (-1)^k k p_{(1^k)} + \sum_{i=1}^{k-1} (-1)^{k-i} i! \left(p_{(1^{k-i})} p_{(i)} \sum_{\sigma \in Sh_i} \sigma \right) = 0, \quad k \le d,$$

$$d! p_{(k)} + (-1)^d d p_{(1^d)} p_{(k-d)} + \sum_{i=1}^{d-1} (-1)^{d-i} i! \left(p_{(1^{d-i})} p_{(k-d+i)} \sum_{\sigma \in Sh_i} \sigma \right) = 0, \quad k > d.$$

TJM - S.B., Drensky, Dzhundrekov, Kassabov

- $k \leq d$, we denote by Sh_i, $i = 0, 1, \ldots, k$, the set of all "shuffles" $\sigma \in \mathrm{Sym}(k)$ with the property that σ^{-1} preserves the orders both of $1, \ldots, k - i$ and of $k - i + 1, \ldots, k$.
- $k > d$ the set Sh_i, $i = 0, 1, \ldots, d$, consists of all permutations $\sigma \in \mathrm{Sym}(k)$, which fix $d + 1, \ldots, k$ and σ^{-1} preserve the orders both of $1, \ldots, d - i$ and of $d - i + 1, \ldots, d$.

We give examples for small $d = 3, 4$.

$$\begin{aligned} 6p_{(3)} &= 3p_{(1,1,1)} - p_{(1,1)}p_{(1)} \circ \big(\mathrm{id} + (321) + (23)\big) \\ &+ 2p_{(1)}p_{(2)} \circ \big(\mathrm{id} + (12) + (123)\big) \end{aligned}$$

$$\begin{aligned} 24p_{(4)} &= -4p_{(1,1,1,1)} + p_{(1,1,1)}p_{(1)} \circ \big(\mathrm{id} + (34) + (432) + (4321)\big) \\ &- 2p_{(1,1)}p_{(2)} \circ \big(\mathrm{id} + (23) + (321) + (13)(24) + (234) + (2134)\big) + \\ &+ 6p_{(1)}p_{(3)} \circ \big(\mathrm{id} + (12) + (123) + (1234)\big) \end{aligned}$$

MDPI - S.B., Drensky, Dzhundrekov, Kassabov

Theorem

When $d \geq \operatorname{char}(K) = p > 0$ the S-algebra $\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)$ is not finitely generated.

Theorem

If $d \geq \operatorname{char}(K) = p > 0$, then the set $\{p_n \mid n = 1, 2, \ldots\}$ is a minimal generating set of the S-algebra $\left(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ\right)$.

Remark

For $d' > d$, we have a projection from $K\langle X_{d'}\rangle$ to $K\langle X_d\rangle$, which sends the extra generators to 0. *It is easy to see that this projection induces a surjective map between the S-algebras of symmetric polynomials. Thus, it is enough to establish that the S-algebra $(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)$ in not finitely generated in the case* $\mathrm{char}\,(K) = p = d$.

- Augmentation ideal $(K\langle X_d\rangle^{\mathtt{Sym}(d)}, \circ)^+$ of $(K\langle X_d\rangle^{\mathtt{Sym}(d)}, \circ)$, i.e., the ideal of polynomials without a constant term in $(K\langle X_d\rangle^{\mathtt{Sym}(d)}, \circ)$.
- Let M_d be the quotient of $(K\langle X_d\rangle^{\mathtt{Sym}(d)}, \circ)^+$ by its square, i.e.,

$$M_d := (K\langle X_d\rangle^{\mathtt{Sym}(d)}, \circ)^+ / \circ \left(\left((K\langle X_d\rangle^{\mathtt{Sym}(d)}, \circ)^+\right)^2\right),$$

  where $\circ(V)$ denotes the submodule of $K\langle X_d\rangle$ generated by V under the action $\circ$.
- $M_d = \bigoplus_{n\in\mathbb{N}} M_d^{(n)}$ is naturally graded and each homogeneous component, $M_d^{(n)}$, is an $\mathtt{Sym}(n)$-module, i.e., there is a natural $\circ$-action on M.

Lemma

The vector space, M_d, is generated as a $\circ$-module and as a vector space by the images of the power sums

$$p_n = x_1^n + \cdots + x_d^n,\ n = 1, 2, \ldots.$$

Consider the abelianization map $\pi : K\langle X_d\rangle \to K[X_d]$ and the map induced by it on the subalgebras of symmetric polynomials. The map, π, is an algebra homomorphism.

Lemma

The map, π, sends a generating set of the S-algebra $(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)$ to a generating set of the commutative algebra $\pi((K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)) \subset K[X_d]^{\mathrm{Sym}(d)}$.

The S-algebra $(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)$ is not finitely generated if we can show that its image under π is not finitely generated.

Remark

Although the map $\pi : K\langle X_d\rangle \to K[X_d]$ *is surjective it does not induce a surjective map between* $K\langle X_d\rangle^{\mathrm{Sym}(d)}$ *and* $K[X_d]^{\mathrm{Sym}(d)}$*.*

For example, if $\mathrm{char}\,(K) = 2$*, then*

$$\pi(x_1x_2 + x_2x_1) = 0 \in K[X_2]^{\mathrm{Sym}(2)}$$

and the elementary symmetric function $e_2 = x_1x_2$ *is not in the image of* π*.*

Let u be a monomial (either in $K\langle X_d\rangle$ or in $K[X_d]$). Since the action of $\mathrm{Sym}(d)$ preserves the set of monomials, one can construct invariants by summing over the orbits of $\mathrm{Sym}(d)$ acting on the set of monomials, i.e.,

$$\sum u = \sum_{g \in \mathrm{Sym}(d)/H_u} g(u)$$

is in the algebra of invariants, where H_u is the stabilizer of the monomial u under the action of the symmetric group $\mathrm{Sym}(d)$.

Lemma

For any monomial $u \in K\langle X_d \rangle$, there exists an integer constant $c_u \in \mathbb{N}$ such that

$$\pi\left(\sum u\right) = c_u\left(\sum \pi(u)\right).$$

Moreover, in the case $p = d$ the constant c_u is 0 in K if and only if $\pi(u) = x_1^s x_2^s \cdots x_p^s$ for some $s \geq 1$.

This is a nontrivial statement because $\sum$ is not an algebraic operation and it is not preserved by the projection π.

For example

$$\sum x_1^2x_2x_3 = x_1^2x_2x_3+x_1^2x_3x_2+x_2^2x_1x_3+x_2^2x_3x_1+x_3^2x_1x_2+x_3^2x_2x_1 \in K\langle X_3\rangle$$

and

$$\sum x_1^2x_2x_3 = x_1^2x_2x_3 + x_2^2x_1x_3 + x_3^2x_1x_2 \in K[X_3].$$

Lemma

In the case $d = p = \mathrm{char}\ K$, *the commutative algebra*

$$\pi(K\langle X_d\rangle^{\mathrm{Sym}(d)}) \subset K[e_1, \dots, e_d] = K[X_d]^{\mathrm{Sym}(d)}$$

is spanned by all products, $e_1^{m_1} \cdots e_d^{m_d}$, *of the elementary symmetric polynomials except the powers,* e_p^m, *of* e_p.

Theorem

When $d \geq \operatorname{char}(K) = p > 0$, the S-algebra $(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)$ is not finitely generated.

Theorem

If $d \geq \operatorname{char}(K) = p > 0$, then the set $\{p_n \mid n = 1, 2, \ldots\}$ is a minimal generating set of the S-algebra $(K\langle X_d\rangle^{\mathrm{Sym}(d)}, \circ)$.

Dihedral invariants - S.B., V. Drensky, S. Findik

Algebra $\mathbb{C}\langle X_2\rangle^{D_{2n}}$ of invariants of the dihedral group D_{2n}.

We assume that the dihedral group

$$D_{2n} = \langle \rho, \tau \,|\, \rho^n = \tau^2 = (\tau\rho)^2 = 1\rangle$$

acts on the free associative algebra $\mathbb{C}\langle u, v\rangle$ as

$$\begin{aligned} \rho : u &\leftarrow \xi u \\ v &\leftarrow \xi^{-1} v \end{aligned} \qquad\qquad \begin{aligned} \tau : u &\leftarrow v \\ v &\leftarrow u \end{aligned}$$

where ξ is the n-th root of unity.

Theorem (Dicks and Formanek, 1982)

The Hilbert series of the algebra of invariants $K\langle X_d\rangle$ of a finite subgroup G of $GL_d(K)$ is

$$H(K\langle X_d\rangle^G) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{1-tr(g)t}$$

In this part, we aim to present a basis, a set of generators of the free algebra $\mathbb{C}\langle u,v\rangle^{D_{2n}}$ and compute its Hilbert series.

Proper monomial and polynomial

Definition - proper monomial

- We call a monomial $w(u,v) \in \mathbb{C}\langle u,v\rangle^{D_{2n}}$ **proper monomial**, if it satisfies the following conditions:
  - ⋆ $w(u,v) = uw'(u,v)$, for some $w'(u,v) \in \mathbb{C}\langle u,v\rangle$
  - ⋆ $\deg_u w - \deg_v w \equiv 0 \pmod n$
  - ⋆ $w(u,v) = uw_1 \dots w_k$, for some $w_i \in u,v$, then

$$\deg_u(uw_1 \dots w_l) - \deg_v(uw_1 \dots w_l) \not\equiv 0 \pmod n$$

for all $l < k$ provided $w_{l+1} = u$.

Definition - proper polynomial

We call a polynomial $w(u,v) + w(v,u)$ **proper polynomial** if $w(u,v)$ is a proper monomial.

Illustration

The monomial $u^4 v = u^3 \cdot uv$ is **not a proper monomial** since

$$\deg_u(u^3) - \deg_v(u^3) \equiv 0 \pmod 3$$
$$\deg_u(uv) - \deg_v(uv) \equiv 0 \pmod 3$$

However, u^3 and uv are **proper monomials** and

$$u^3 + v^3, \quad uv + vu$$

are **proper polynomials.**

Dihedral invariants

Theorem

(i) *The vector space* $\mathbb{C}\langle u, v\rangle^{D_{2n}}$ *is of basis consisting of* 1 *and elements of the form*

$$w(u, v) + w(v, u)$$

such that $w(u, v)$ *is a product of proper monomials.*

(ii) *The free algebra* $\mathbb{C}\langle u, v\rangle^{D_{2n}}$ *is generated by proper polynomials.*

Let $h_{2n}(t)$ and $g_{2n}(t)$ be the Hilbert series and the generating function of the free algebra $\mathbb{C}\langle u,v\rangle^{D_{2n}}$, respectively.

Theorem

(i) If $n=2m+1$, $m\geq 1$, then

$$h_{2n}(t)=\frac{1}{2}+\frac{1}{2n(1-2t)}+\frac{1}{n}\sum_{k=1}^{m}\frac{1}{1-2\cos(\frac{2k\pi}{n})t}$$

(ii) If $n=2m+2$, $m\geq 1$, then

$$h_{2n}(t)=\frac{1}{2}+\frac{1}{2n(1-2t)}+\frac{1}{2n(1+2t)}+\frac{1}{n}\sum_{k=1}^{m}\frac{1}{1-2\cos(\frac{2k\pi}{n})t}$$

Theorem

The S-algebra $\mathbb{C}\langle u, v\rangle^{D_{2n}}$ is generated (as an S-algebra) by $uv + vu$ and $u^n + v^n$.

Theorem

S-algebra $\mathbb{C}\langle u, v\rangle^{D_{2n}}$ is spanned by

$$s_a \circ \sigma = (u^a v^a + v^a u^a) \circ \sigma, \quad a = 1, 2, \ldots, \sigma \in \mathrm{Sym}_{2a},$$

$$p_{(b,c)} \circ \tau = (u^b v^c + v^b u^c) \circ \tau, \quad b - c \equiv 0 \pmod{n}, \tau \in \mathrm{Sym}_{b+c},$$

i.e. it is generated as an S-algebra by

$$s_a = u^a v^a + v^a u^a, \quad a = 1, 2, \ldots,$$

$$p_{(b,c)} = u^b v^c + v^b u^c, \quad b - c \equiv 0 \pmod{n}.$$

Noncommutative alternative polynomials

Any noncommutative alternative polynomial $f \in K\langle X_d\rangle^{\mathrm{Alt}(d)}$ can be written as

$$f = f_1 + f_2,$$

where f_1 is symmetric polynomial in d non commuting variables and f_2 is alternating, i.e. f_2 changes sign whenever we exchange any two variables.

Since the algebra of alternative noncommutative polynomials contains the algebra of symmetric noncommutative ones, this allows us to only study alternating polynomaials.

If $u \in \langle X_d \rangle$ is a monomial in d noncommuting variables, by $\sum\limits_{\mathrm{Alt}} u$ we denote the alternating sum

$$\sum_{\sigma \in \mathrm{Sym}(d)} (-1)^{\sigma} u^{\sigma}.$$

The S-algebra of the noncommutative alternative polynomials is generated by the elementary symmetric polynomials $p_{(1^i)}$, $i = 1, \cdots, d$, together with the polynomials $s_k = \sum\limits_{\mathrm{Alt}} x_1^{k-1} x_2$, $k = 1, 2, \ldots$.

We shrink the generating set $\{p_{(1^i)} \mid i = 1, \cdots, d\} \cup \{s_k \mid k \in \mathbb{N}^+\}$ to a finite set.

Theorem

Let $\mathrm{char}(\mathrm{K}) = 0$ *or* $\mathrm{char}(\mathrm{K}) = p > 3$. *Then the* S*-algebra of the alternative polynomials in* 3 *noncommuting variables* $\left(K\langle X_3\rangle^{\mathrm{Alt}(3)}, \circ\right)$ *is generated as an* S*-algebra by the elementary symmetric polynomials* $p_{(1^i)}, i = 1, 2, 3,$ *together with the alternating polynomials* $s_2 = \sum\limits_{\mathrm{Alt}} x_1 x_2$ *and* $s_3 = \sum\limits_{\mathrm{Alt}} x_1^2 x_2$.

Theorem.

The S*-algebra* $\left(K\langle X_3\rangle^{\mathrm{Alt}(3)}, \circ\right)$ *is not finitely generated for fields* K *of characteristic* 2 *or* 3.

Conjecture

Let $\mathrm{char}(\mathrm{K}) = 0$ *or* $\mathrm{char}(\mathrm{K}) = p > d$. *Then the* S*-algebra of the alternative polynomials in* d *noncommuting variables* $\left(K\langle X_d\rangle^{\mathrm{Alt}(d)}, \circ\right)$ *is generated as an* S*-algebra by the elementary symmetric polynomials* $p_{(1^i)}, i = 1, \cdots, d$, *together with the alternating polynomials* $s_{d-1} = \sum_{\mathrm{Alt}} x_1^{d-2} x_2$ *and* $s_d = \sum_{\mathrm{Alt}} x_1^{d-1} x_2$.

THANK YOU FOR ATTENTION!